

# GEOMETRY OF HOUGHTON'S GROUPS

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ABSTRACT. Houghton's groups  $\{\mathcal{H}_n\}$  is a family of groups where each  $\mathcal{H}_n$  consists of 'translations at infinity' on  $n$  rays of discrete points emanating from the origin on the plane. Brown shows  $\mathcal{H}_n$  has type  $FP_{n-1}$  but not  $FP_n$  by constructing infinite dimensional cell complex on which  $\mathcal{H}_n$  acts with certain conditions. We modify his idea to construct  $n$ -dimensional CAT(0) cubical complex  $X_n$  on which  $\mathcal{H}_n$  acts with the same conditions as before. Brown also shows  $\mathcal{H}_n$  is finitely presented provided  $n \geq 3$ . Johnson provides a finite presentation for  $\mathcal{H}_3$ . We extend his result to provide finite presentations of  $\mathcal{H}_n$  for  $n > 3$ . We also establish exponential isoperimetric inequalities of  $\mathcal{H}_n$  for  $n \geq 3$ .

## 1. INTRODUCTION

One of major paradigms in geometric group theory is the idea that one can understand algebraic properties of groups by studying their actions on geometric spaces. There is a basic geometric object that one can associate to any finitely generated group: its Cayley graph with the word metric.

Every pair  $(G, S)$  of a group and its finite generating set  $S$  has an associated *word metric*. The distance  $d(g, h)$  between elements  $g, h \in G$  is defined to be the length of the shortest word in  $S \cup S^{-1}$  which is equal to  $g^{-1}h$  in  $G$ . The *Cayley graph*  $\Gamma(G, S)$  of a pair  $(G, S)$  has  $G$  as its vertex set, and a vertex  $g_1$  is connected to another vertex  $g_2$  by an edge labeled by  $s$  if  $g_2 = g_1s$  for some  $s \in S$ . A word metric on a pair  $(G, S)$  extends to a metric on  $\Gamma(G, S)$  provided one declares each edge in  $\Gamma(G, S)$  has length 1. The action of  $G$  on itself by left multiplication extends naturally to an isometric action on the Cayley graph (with fixed generating set  $S$ ).

We can use a Cayley graph  $\Gamma(G, S)$  to define geometric properties of a group  $G$ . One such property is the number of ends  $e(G)$  of  $G$ . Roughly speaking, the number of ends  $e(G)$  of a finitely generated group  $G$  measures the number of "connected components of  $G$  at infinity". For a finitely generated group  $G$ ,  $e(G)$  is determined by the number of *ends*  $e(\Gamma)$ , of a Cayley graph  $\Gamma$  of  $G$ , which can be defined as follows. To find  $e(\Gamma)$ , remove a compact set  $K$  from  $\Gamma$ , and count the number of unbounded components of  $\Gamma - K$ . The number  $e(\Gamma)$  is defined to be the supremum of this number over all compact sets. Finally it can be shown that  $e(\Gamma)$  is independent of the choice of a finite generating set  $S$  used in the construction of  $\Gamma$  ([20]). So it makes sense to define  $e(G) = e(\Gamma)$ .

In [19], Stallings showed that, for a finitely generated group  $G$ , the geometric condition of having more than one end is equivalent to the algebraic condition that  $G$  *splits* (that is,  $G$  can be written as a free product with amalgamation or HNN extension) over a finite subgroup. On the other hand, the theory of Bass-Serre relates the algebraic condition that  $G$  splits to certain types of actions of  $G$  on a tree. In view of Bass-Serre theory, Stallings' theorem can be restated as follows.

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**Theorem 1.1.** *A finitely generated group  $G$  satisfies  $e(G) > 1$  if and only if  $G$  acts on a tree (without inversion) with finite edge-stabilizers.*

There is a generalization of Theorem 1.1, which replaces the tree by a possibly higher dimensional space and uses a more general notion of ends than  $e(G)$ . In [11], C. H. Houghton introduced the concept of the number of ends  $e(G, H)$  of a finitely generated group with respect to a subgroup  $H \leq G$ . Being a subgroup of  $G$ ,  $H$  acts on a Cayley graph  $\Gamma$  of  $G$  by the left multiplication. Now define  $e(G, H)$  to be the number of ends of the quotient graph  $\Gamma/H$ . Again, one can show that this is independent of the particular finite generating set  $S$  used in the construction of  $\Gamma$  ([18]). A group  $G$  is called *multi-ended* if  $e(G, H) > 1$  for some subgroup  $H$ .

Roughly speaking, a piecewise Euclidean (PE) cubical complex is build from a collection of a regular Euclidean cubes by glueing their faces via isometries. A *cubing* is a 1-connected PE cubical complex satisfying some additional non-positive curvature conditions. We will make this notion precise in Section 3.3.

One dimensional cubes are just unit length line segments, and so a 1-dimensional cubical complex is simply a graph. The condition of being simply connected means that the graph is a tree. Therefore one can think of cubings as generalizations of trees.

In [17], Sageev proved the following remarkable generalization of Stallings' result.

**Theorem 1.2.** *A finitely generated group  $G$  is multi-ended if and only if  $G$  acts 'essentially' on a cubing.*

This theorem includes the possibility that the cubing is infinite dimensional. An essential action means that the action has an unbounded orbit provided the cubing is finite dimensional.

Non-positively curved cubical complexes play a central role in low-dimensional topology and geometric group theory. A striking example of their importance is given by Agol's recent proof [1] of the Virtual Haken Conjecture and the Virtual Fibration Conjecture in 3-manifold topology. Agol's proof relies on results of Haglund and Wise which concern fundamental groups of a *speical* class of non-positively curved cubical complexes. In [10], Haglund and Wise showed if the fundamental group of a special cubical complex is word-hyperbolic then every quasiconvex subgroup is separable.

This thesis explores geometric properties of a particular class of groups, termed Houghton's groups, introduced in [12]. Roughly speaking, Houghton's group  $\mathcal{H}_n$  ( $n \in \mathbb{N}$ ) is the group of permutations of  $n$  rays of discrete points which are eventual translations (each permutation acts as a translation along each ray outside a finite set). See Section 2.1 for details.

There are  $n$  canonical copies of  $\mathcal{H}_{n-1}$  inside  $\mathcal{H}_n$ , and  $\mathcal{H}_n$  is multi-ended with respect to each of them. The  $i^{th}$  subgroup,  $1 \leq i \leq n$ , is obtained by restricting to permutations which fix  $i^{th}$  ray pointwise. One of the main results of this thesis is to produce an action of  $\mathcal{H}_n$  on a  $n$ -dimensional cubing  $X_n$ . Note that depending on subgroups which are taken into account, there are various cubings on which  $\mathcal{H}_n$  acts. One feature of our cubing is that  $X_n$  encodes all of those subgroups  $\mathcal{H}_{n-1}$  at once. See Section ?? for a detailed description.

**Theorem A.** *For each integer  $n \geq 1$ , there exists a  $n$ -dimensional cubing  $X_n$  and a Morse function  $h : X_n \rightarrow \mathbb{R}_{\geq 0}$  such that  $\mathcal{H}_n$  acts on  $X_n$  properly (but not cocompactly) by height-preserving semi-simple isometries. Furthermore, for each  $r \in \mathbb{R}_{\geq 0}$  the action of  $\mathcal{H}_n$  restricted to the level set  $h^{-1}(r)$  is cocompact.*

An additional feature of  $X_n$  is that it comes equipped with a height function (Morse function) to the non-negative reals. The group  $\mathcal{H}_n$  acts as a height-preserving fashion, and the quotient of any level set by  $\mathcal{H}_n$  is cocompact. As an application we recover Brown's results for finiteness properties of  $\mathcal{H}_n$ .

**Corollary B.** *For  $n \geq 2$ ,  $\mathcal{H}_n$  is of type  $FP_{n-1}$  but not  $FP_n$ , it is finitely presented for  $n \geq 3$ .*

Knowing that  $\mathcal{H}_n$  is finitely presented for  $n \geq 3$  prompts a natural question. What are explicit presentations for the  $\mathcal{H}_n$ ? In [14], Johnson answered this for  $\mathcal{H}_3$ . Another main result of this thesis provides explicit presentations for all  $\mathcal{H}_n$  ( $n \geq 3$ ).

**Theorem C.** *For  $n \geq 3$ ,  $\mathcal{H}_n$  is generated by  $g_1, \dots, g_{n-1}, \alpha$  with relators*

$$\alpha^2 = 1, (\alpha \alpha^{g_1})^3 = 1, [\alpha, \alpha^{g_i}] = 1, \alpha = [g_i, g_j], \alpha^{g_i^{-1}} = \alpha^{g_i^{-1}} \text{ for } 1 \leq i \neq j \leq n-1.$$

Finally we determine bounds for the Dehn functions of  $\mathcal{H}_n$ . We give a formal definition of Dehn functions in Section 4. Intuitively, given a finite presentation  $P = \langle A | R \rangle$  for a group  $G$ , and given a word  $w$  representing the identity in  $G$  with  $|w| \leq x$ , the Dehn function of a presentation  $P$ ,  $\delta_P(x)$ , measures the least upper bound on the number of relations in term of  $x$ , which one must apply to check  $w = 1$ . Although the function  $\delta_P(x)$  depends on the presentation, the growth type of this function is independent of choice of a finite presentation for  $G$ . The *Dehn function* of a finitely presented group  $G$  is defined to be the growth type of  $\delta_P(x)$ . See Section 4.1 for details. An isoperimetric function for a group is an upper bound of the Dehn function. In Section 4 we establish isoperimetric inequalities for  $\mathcal{H}_n$  for  $n \geq 3$ .

**Theorem D.** *For any  $n \geq 3$  the Dehn function  $\delta_{\mathcal{H}_n}(x)$  of  $\mathcal{H}_n$  satisfies*

$$\delta_{\mathcal{H}_n}(x) \preceq e^x.$$

## 2. HOUGHTON'S GROUPS $\mathcal{H}_n$

**2.1. Definition of  $\mathcal{H}_n$ .** Fix an integer  $n \geq 1$ . Let  $\mathbb{N}$  be the positive integers. For each  $k$ ,  $1 \leq k \leq n$ , let

$$R_k = \{me^{i\theta} : m \in \mathbb{N}, \theta = \pi/2 + 2\pi(k-1)/n\} \subset \mathbb{C}.$$

and let  $Y_n = \bigcup_{k=1}^n R_k$  be the disjoint union of  $n$  copies of  $\mathbb{N}$ , each arranged along a ray emanating from the origin in the plane. We shall use the notation  $\{1, \dots, n\} \times \mathbb{N}$  for  $Y_n$ , letting  $(k, p)$  denote the point of  $R_k$  with distance  $p$  from the origin. A bijection  $g : Y_n \rightarrow Y_n$  is called an *eventual translation* if it acts as a translation on each  $R_k$  outside a finite set. More precisely  $g$  is an eventual translation if the following holds:

- ★ There is an  $n$ -tuple  $(m_1, \dots, m_n) \in \mathbb{Z}^n$  and a finite set  $K_g \subset Y_n$  such that  $(k, p) \cdot g = (k, p + m_k)$  for all  $(k, p) \in Y_n - K_g$ .

**Definition 2.1.** For an integer  $n \geq 1$ , Houghton's group  $\mathcal{H}_n$  is defined to be the group of all eventual translations of  $Y_n$ .

As indicated in the definition ★, Houghton's group  $\mathcal{H}_n$  acts on  $Y_n$  on the right; thus  $gh$  denotes  $g$  followed by  $h$  for  $g, h \in \mathcal{H}_n$ . For notational convenience we denote  $g^{-1}$  by  $\bar{g}$ .

Let  $g_i$  be the translation on the ray of  $R_1 \cup R_{i+1}$ , from  $R_1$  to  $R_{i+1}$  by 1 for  $1 \leq i \leq n-1$ . More precisely,  $g_i$  is defined by

$$(2.1) \quad (j, p)g_i = \begin{cases} (1, p-1) & \text{if } j = 1 \text{ and } p \geq 2, \\ (i+1, 1) & \text{if } (j, p) = (1, 1) \\ (i+1, p+1) & \text{if } j = i+1, \\ (j, p) & \text{otherwise.} \end{cases}$$

Figure 2.1 illustrates some examples of elements of  $\mathcal{H}_n$ , where points which do not involve arrows are meant to be fixed, and points of each finite set  $K$  are indicated by circles. Finite sets  $K_{g_i}$  and  $K_{g_j}$  are singleton sets. After simple computation, one can check that the commutator of  $g_i$  and  $g_j$  ( $i \neq j$ ) is the transposition exchanging  $(1, 1)$  and  $(1, 2)$ . The last element  $g$  is rather generic and  $K_g$  consists of eight points (circles).

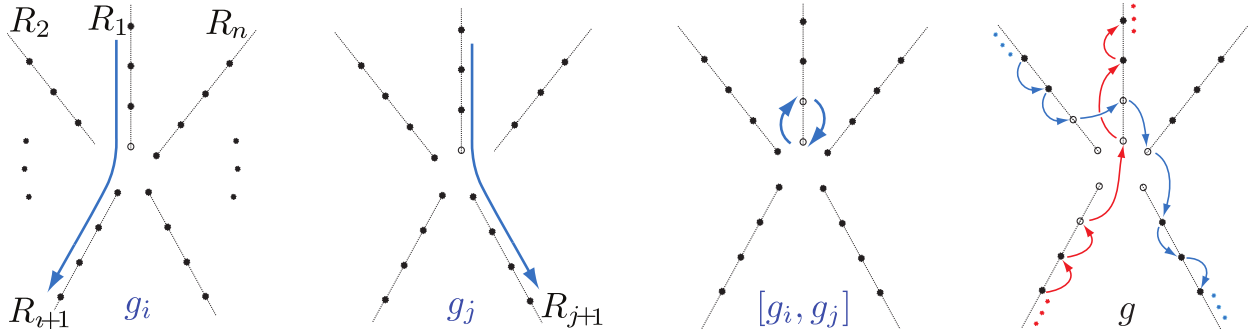


FIGURE 2.1. Some elements of  $\mathcal{H}_n$

**2.2. Abelianization of  $\mathcal{H}_n$ .** Assigning an  $n$ -tuple  $(m_1, \dots, m_n) \in \mathbb{Z}^n$  to each  $g \in \mathcal{H}_n$  defines a homomorphism  $\varphi : \mathcal{H}_n \rightarrow \mathbb{Z}^n$ . Let  $\Sigma_{n,\infty}$  denote the infinite symmetric group consisting of all permutations of  $Y_n$  with finite support.

**Lemma 2.2.** For  $n \in \mathbb{N}$ ,  $\Sigma_{n,\infty} \leq \mathcal{H}_n$ .

*Proof.* Every element  $f \in \Sigma_{n,\infty}$  has a finite support. This means there exists a finite set  $K \subset Y_n$  such that  $f$  acts on each  $R_i \subset Y_n$  as the identity outside  $K$ . So  $f \in \mathcal{H}_n$ .  $\square$

**Lemma 2.3.** For  $n \geq 3$ ,  $\ker \varphi = \Sigma_{n,\infty} = [\mathcal{H}_n, \mathcal{H}_n]$ .

*Proof.* Fix  $n \geq 3$ . We show the equalities by verifying a chain of inclusions. Each element  $g \in \ker \varphi$  acts on  $Y_n$  as the identity outside a finite set  $K \subset Y_n$ . This means that  $g$  is a permutation on  $K$ , and so  $g \in \Sigma_{n,\infty}$ . The claim in the proof of Lemma 2.7 implies that every element with finite support is a product of conjugations of  $[g_i, g_j]$  ( $i \neq j$ ). Suppose  $g \in [\mathcal{H}_n, \mathcal{H}_n]$ . Lemma 2.7 says  $\mathcal{H}_n$  is generated by  $g_1, \dots, g_{n-1}$  if  $n \geq 3$ . Take an expression for  $g$  in letters  $g_1, \dots, g_{n-1}$ . Since  $g \in [\mathcal{H}_n, \mathcal{H}_n]$ , the sum of powers of  $g_i$ 's in this expression is 0 for each  $i = 1, \dots, n-1$ . So  $g \in \ker \varphi$ .  $\square$

In Section 2.3, we shall see that  $\mathcal{H}_n$  is generated by  $g_1, \dots, g_{n-1}$  defined in equation (2.1). Note that  $\varphi(g_i) \in \mathbb{Z}^n$  has only two nonzero values  $-1, 1$ , and

$$\varphi(g_i) = (-1, 0, \dots, 0, 1, 0, \dots, 0)$$

where 1 occurs in  $(i+1)^{th}$  component. The image of  $\mathcal{H}_n$  is generated by those elements and  $\varphi(\mathcal{H}_n) = \{(m_1, \dots, m_n) \mid \sum m_i = 0\}$  is free abelian of rank  $n-1$ .

**Corollary 2.4.** *For  $n \geq 3$ , the abelianization of  $\mathcal{H}_n$  is given by the following short exact sequence.*

$$(2.2) \quad 1 \rightarrow [\mathcal{H}_n, \mathcal{H}_n] \rightarrow \mathcal{H}_n \xrightarrow{\varphi} \mathbb{Z}^{n-1} \rightarrow 1.$$

The above result was originally shown by C. H. Houghton in [12]. Note that the group  $\mathcal{H}_n$  has the property that the rank of the abelianization is one less than the number of rays of the space  $Y_n$  (namely, the number of ends of  $Y_n$  with respect to the action) on which  $\mathcal{H}_n$  acts transitively. This was the main reason that Houghton introduced and studied a family of groups  $\{\mathcal{H}_n\}_{n \in \mathbb{N}}$  in the same paper.

**Remark 2.5.** For  $n \geq 3$ , by replacing the commutator subgroup by  $\Sigma_{n,\infty}$  in the short exact sequence (2.2), we have the following short exact sequence.

$$(2.3) \quad 1 \rightarrow \Sigma_{n,\infty} \rightarrow \mathcal{H}_n \xrightarrow{\varphi} \mathbb{Z}^{n-1} \rightarrow 1,$$

The embedding  $\Sigma_{n,\infty} \hookrightarrow \mathcal{H}_n$  is crucial for our study in two directions. Firstly, it gives rise to finite presentation for  $\mathcal{H}_n$  for  $n \geq 3$  (Section 2.3). Secondly, this inclusion is used to construct exponential isoperimetric inequalities for  $\mathcal{H}_n$   $n \geq 3$  (Section 4.2).

We suppress  $n$  in  $\Sigma_{n,\infty}$  when the underlying set  $Y_n$  is clear from the context.

**2.3. Finite Presentations for  $\mathcal{H}_n$  ( $n \geq 3$ ).** Note that every element of  $\mathcal{H}_1$  acts on  $Y_1 = \mathbb{N}$  by the identity outside a finite set. If the element acted as a non-trivial translation outside of a finite set, then it would fail to be a bijection of  $Y_1$ . So  $\mathcal{H}_1 \subset \Sigma_{1,\infty}$ . The converse inclusion is obvious, therefore  $\mathcal{H}_1$  is the infinite symmetric group  $\Sigma_{1,\infty}$  of all permutations of  $\mathbb{N}$  with finite support. By considering elements with successively larger supports, one can argue that  $\Sigma_{1,\infty}$  is not finitely generated. It is known [6] that  $\mathcal{H}_2$  is finitely generated but not finitely presented, and that  $\mathcal{H}_n$  is finitely presented for  $n \geq 3$ . Brown showed more than just finite presenteness, and in Section 3.7, we will give a precise statement and a new proof of Brown's result about the finiteness properties of  $\mathcal{H}_n$  (Corollary B). In this section we will provide explicit finite presentations for  $\mathcal{H}_n$  ( $n \geq 3$ ).

**Finite generating sets for  $\mathcal{H}_n$  ( $n \geq 2$ ).**

We begin by proving that  $\mathcal{H}_2$  is generated by two elements. Many of the techniques in this argument extend to the  $\mathcal{H}_n$  for  $n \geq 3$ . Consider the two elements  $g_1, \beta \in \mathcal{H}_2$  where  $g_1$  is defined by equation (2.1) and  $\beta$  is the transposition exchanging  $(1, 1)$  and  $(2, 1)$  as depicted in Figure 2.2.



FIGURE 2.2.  $\mathcal{H}_2$  is generated by  $g_1$  and  $\beta$

Suppose  $g \in \mathcal{H}_2$ . Since  $\varphi(g_1)$  generates the image  $\varphi(\mathcal{H}_2) \cong \mathbb{Z}$  there exists  $k \in \mathbb{Z}$  such that the given  $g$  and  $g_1^k$  agree on  $Y_2 - K$  for some finite set  $K \subset Y_2$ . So we have

$$\bar{g}_1^k g = f \quad \text{or} \quad g = g_1^k f$$

where  $f \in \text{Perm}(K)$ . One can take suitable conjugates  $\bar{g}_1^{k'} \beta g_1^{k'}$  to express transpositions exchanging two consecutive points of  $Y_2$ . For example,  $\bar{g}_1^3 \beta g_1^3$  is the transposition swapping  $(2, 3)$  and  $(2, 4)$ . Since any permutation of a finite set  $K$  can be written as a product of those transpositions,  $f$  can be written as a product of conjugations of  $\beta$ . So  $g = g_1^k f \in \mathcal{H}_2$ .

**Remark 2.6.** The conjugation  $\bar{h}gh$  is denoted by  $g^h$ . Note that  $(i, p)g = (j, q)$  is equivalent to  $(i, p)hg^h = (j, q)h$ . The following observation is quite elementary but useful: If  $\beta$  is a transposition exchanging two points  $(i, p)$  and  $(j, q)$  then  $\beta^h$  is the transposition exchanging  $(i, p)h$  and  $(j, q)h$ .

We have seen that a translation and a transposition are sufficient to generate all of  $\mathcal{H}_2$ . Moreover, we have seen from the third example of Figure 2.1 that  $[g_i, g_j]$  is a transposition on  $Y_n$  for  $1 \leq i \neq j \leq n-1$ . So the following lemma should not be a surprise.

**Lemma 2.7.**  $\mathcal{H}_n$  is generated by  $g_1, \dots, g_{n-1}$  for  $n \geq 3$ .

*Proof.* Let  $g \in \mathcal{H}_n$  and  $\varphi(g) = (m_1, \dots, m_n)$ . Note that  $m_1 = -(m_2 + \dots + m_n)$ . Consider the element  $g' \in \mathcal{H}_n$  given by

$$g' = g_1^{m_2} g_2^{m_3} \dots g_{n-1}^{m_n}.$$

By construction,  $\varphi(g') = \varphi(g)$ , and so two element  $g'$  and  $g$  agree on  $Y_n - K$  for some finite set  $K$ . This means there exists  $f \in \text{Perm}(K)$  such that  $g = g'f$ . So it suffices to show that every finite permutation can be written as a product of the  $g_i^{\pm 1}$ ,  $i = 1, \dots, n-1$ .

**Claim:** Any transposition of  $Y_n$  is a conjugation of  $[g_1, g_2] = g_1 g_2 \bar{g}_1 \bar{g}_2$  by a product of  $g_i$ 's. First we express transpositions exchanging points of  $R_1$  in this manner. We already saw that  $\alpha = [g_1, g_2]$  is the transposition exchanging  $(1, 1)$  and  $(1, 2)$ .

Consider the following elements  $h_1, h_2 \in \mathcal{H}_n$ :

$$h_1 = \alpha^{g_1 \bar{g}_2^{(p-2)} \bar{g}_1} \text{ and } h_2 = \alpha^{g_1 \bar{g}_2^{(q-2)} \bar{g}_1}$$

where  $p, q \geq 2$  are integers. By the observation in Remark 2.6,  $h_1$  is the transposition exchanging  $(1, 1)$  and  $(1, p)$ . By the same reason, one sees  $h_2$  is the transposition swapping  $(1, 1)$  and  $(1, q)$ . Therefore the permutation exchanging  $(1, p)$  and  $(1, q)$  can be expresses by the conjugation  $h_1^{h_2}$ . This allows one to express transpositions of  $R_{i+1}$  for all  $i$  by conjugations  $h_1^{h_2 g_i^m}$  for some  $m \in \mathbb{N}$ .

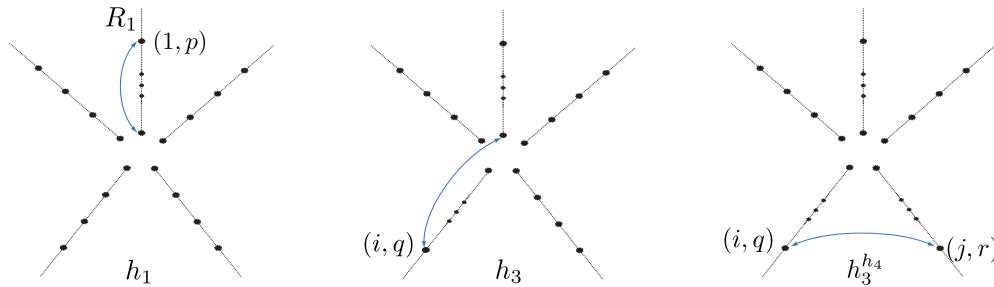


FIGURE 2.3. Transpositions are conjugations of  $[g_i, g_j]$

Next we show that transpositions exchanging points of different rays can be expressed in a similar fashion. Set  $2 \leq i \neq j \leq n$  and  $q, r \in \mathbb{N}$ . The permutations  $h_3, h_4$  given by

$$h_3 = \alpha^{g_j g_i^q \bar{g}_j} \text{ and } h_4 = \alpha^{g_i g_j^r \bar{g}_i}$$

are the transpositions swapping  $(1, 1)$  and  $(i, q)$  and swapping  $(1, 1)$  and  $(j, r)$  respectively. Thus the conjugation  $h_1^{h_3}$  represents the transposition exchanging  $(1, p)$  and  $(i, q)$ . Similarly the conjugation  $h_3^{h_4}$  represents the transposition swapping  $(i, q)$  and  $(j, r)$ . Figure 2.3 illustrates various transpositions of  $Y_n$  given by conjugations of  $\alpha$ . Our claim follows since  $i, j, p, q$  and  $r$  are arbitrary. So we are done because any finite permutation  $f \in \text{Perm}(K)$  above can be written as a product of transpositions.  $\square$

### Finite presentation for $\mathcal{H}_3$ .

D. L. Johnson ([14]) provided a finite presentation for  $\mathcal{H}_3$ .

**Theorem 2.8.** *The Houghton's group  $\mathcal{H}_3$  is isomorphic to  $H_3$  presented by*

$$(2.4) \quad H_3 = \langle g_1, g_2, \alpha \mid r_1, \dots, r_5 \rangle,$$

$$\begin{aligned} r_1 &: \alpha^2 = 1, \\ r_2 &: (\alpha \alpha^{\bar{g}_1})^3 = 1, \\ r_3 &: [\alpha, \alpha^{\bar{g}_1}] = 1, \\ r_4 &: [g_1, g_2] = \alpha, \\ r_5 &: \alpha^{\bar{g}_1} = \alpha^{\bar{g}_2}. \end{aligned}$$

Note that there is a map  $\psi$  from an abstract group  $H_3$  to  $\mathcal{H}_3$  such that  $\psi(g_i)$  is the translation  $g_i$  defined in equation (2.1) ( $i = 1, 2$ ) and  $\psi(\alpha)$  is the transposition  $\alpha = [g_1, g_2]$ . Observe that relations  $r_1, r_2$  and  $r_3$  are reminiscent relations of Coxeter systems for finite symmetric groups. It is easy to check that  $\mathcal{H}_3$  satisfies those five relations. Johnson showed that the group  $H_3$  fits into the short exact sequence of (2.3) with the normal subgroup generated by  $\alpha$ , and that  $\psi : H_3 \rightarrow \mathcal{H}_3$  induces isomorphisms between the normal subgroups and the quotients. So the following diagram commute. The Five Lemma completes the proof.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle\langle \alpha \rangle\rangle & \longrightarrow & H_3 & \longrightarrow & H_3 / \langle\langle \alpha \rangle\rangle & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & [\mathcal{H}_3, \mathcal{H}_3] & \longrightarrow & \mathcal{H}_3 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 1. \end{array}$$

### Finite presentations for $\mathcal{H}_n$ ( $n \geq 3$ ).

We extend the previous argument to find finite presentations for  $\mathcal{H}_n$  for all  $n \geq 3$ . The first step is to establish appropriate presentations for symmetric groups on finite balls of  $Y_n$  by applying Tietze transformation to the Coxeter systems for finite symmetric groups.

Fix positive integers  $n, r \in \mathbb{N}$ . Let  $B_{n,r}$  denote the ball of  $Y_n$  centered at the origin with radius  $r$ , i.e.,  $B_{n,r} = \{(i, p) \in Y_n : 1 \leq i \leq n, 1 \leq p \leq r\}$ . The ball  $B_{n,r}$  contains  $nr$  points which can be identified to the points of  $\{1, 2, \dots, r, \dots, 2r, \dots, nr\} \subset \mathbb{N}$  via  $\chi : B_{n,r} \rightarrow \mathbb{N}$  defined by

$$\chi(i, p) = (i - 1)r + p.$$

Recall the following Coxeter system for the symmetric group  $S_{nr}$  on  $\{1, 2, \dots, nr\}$

$$(2.5) \quad S_{nr} = \langle \sigma_1, \dots, \sigma_{nr-1} \mid \sigma_k^2, (\sigma_k \sigma_{k+1})^3, [\sigma_k, \sigma_{k'}] \text{ for } |k - k'| \geq 2 \rangle$$

**Remark 2.9.** One can interpret the above three types of relators respectively as follows;

- generators are involutions,

- two generators with overlapping support satisfy braid relations,
- two generators commute if they have disjoint supports.

Note that every collection of transpositions of  $S_{nr}$  satisfies the three types of relations. We shall see in Theorem 2.10 that a ‘reasonable’ set of transpositions satisfying the above relations provides a presentation for the symmetric group  $S_{nr}$ .

The arrangement of  $Y_n$  in the plane is different than the arrangement of  $nr$  points  $\{1, 2, \dots, nr\}$  in  $\mathbb{N}$ . In particular, the adjacency is different. By a pair of two *adjacent* points in  $Y_n$ , we mean either a pair of two consecutive points in a ray of  $Y_n$  or a pair of  $(1, 1)$  and  $(j+1, 1)$  for  $1 \leq j \leq n-1$ . We aim to produce a new presentation for the symmetric group  $\Sigma_{n,r}$  on  $B_{n,r}$  with a generating set which consists of the transpositions exchanging all pairs of adjacent points in  $Y_n$ . More precisely we need the following  $nr-1$  transpositions

$$\begin{aligned} \alpha_p^i, & \text{ exchanging } (i, p) \text{ and } (i, p+1) \text{ for } 1 \leq i \leq n \text{ and } 1 \leq p \leq r-1 \\ \alpha_0^{j+1}, & \text{ exchanging } (1, 1) \text{ and } (j+1, 1) \text{ for } 1 \leq j \leq n-1. \end{aligned}$$

For an element  $\sigma \in S_{nr}$  consider the element of  $Perm(B_{n,r})$  given by

$$(\sigma)^{\chi^{-1}} = \chi(\sigma)\chi^{-1}$$

which stands for the composition  $B_{n,r} \xrightarrow{\chi} \{1, \dots, nr\} \xrightarrow{\sigma} \{1, \dots, nr\} \xrightarrow{\chi^{-1}} B_{n,r}$ . Let us denote the induced element by  $\chi^*(\sigma)$ . New generators of the first type are defined by  $\chi^*$  as follows.

$$(2.6) \quad \alpha_p^i := \chi^*(\sigma_{(i-1)r+p})$$

for  $1 \leq i \leq n$  and  $1 \leq p \leq r-1$ . Note  $\chi^*$  defines a bijection

$$\{\sigma_1, \dots, \widehat{\sigma_r}, \dots, \widehat{\sigma_{2r}}, \dots, \widehat{\sigma_{(n-1)r}}, \dots, \sigma_{nr-1}\} \leftrightarrow \{\alpha_p^i \mid 1 \leq i \leq n, 1 \leq p \leq r-1\},$$

( $\widehat{\phantom{x}}$  denotes deletion).

Replace  $\sigma_{(i-1)r+p}$  by  $\alpha_p^i$  in the Coxeter system (2.5),  $1 \leq i \leq n$  and  $1 \leq p \leq r-1$ . Then (2.5) becomes

$$(2.7) \quad S_{nr} = \langle \alpha_p^i, \sigma_{jr} \mid R \rangle \quad 1 \leq i \leq n, 1 \leq p \leq r-1, \text{ and } 1 \leq j \leq n-1,$$

where  $R$  consisting of

$$\begin{aligned} R_1: & (\alpha_p^i)^2, (\sigma_{jr})^2, \forall i, p, j \\ R_2: & [\alpha_p^i, \alpha_q^i], \forall i, |p-q| \geq 2 \\ R_3: & [\alpha_p^i, \alpha_{p'}^i], \forall p, q, i \neq i' \\ R_4: & [\alpha_p^i, \sigma_{jr}], \forall j, (i-1)r+p \neq jr \pm 1 \\ R_5: & [\sigma_{jr}, \sigma_{j'r}], j \neq j' \\ R_6: & (\alpha_p^i \alpha_{p+1}^i)^3, \forall i, 1 \leq p \leq r-2 \\ R_7: & (\alpha_p^i \sigma_{jr})^3, \forall j, (i-1)r+p = jr \pm 1. \end{aligned}$$

Note that the difference in presentations (2.5) and (2.7) is just notation and the set  $R$  of relations exhibits the same idea of Remark 2.9. We also want to replace  $\sigma_{jr}$  by  $\alpha_0^{j+1}$  for  $j = 1, \dots, n-1$  as follows. For each  $j$ ,  $\alpha_0^{j+1}$  corresponds, under  $\chi^*$ , to the transposition  $(1 \ jr+1)$  of  $S_{nr}$ . Observe that the transposition  $(1 \ jr+1)$  of  $S_{nr}$  can be expressed as  $(\sigma_{jr})^{\overline{w_{jr-1}}}$  where  $w_k = \sigma_1 \sigma_2 \dots \sigma_k$ .



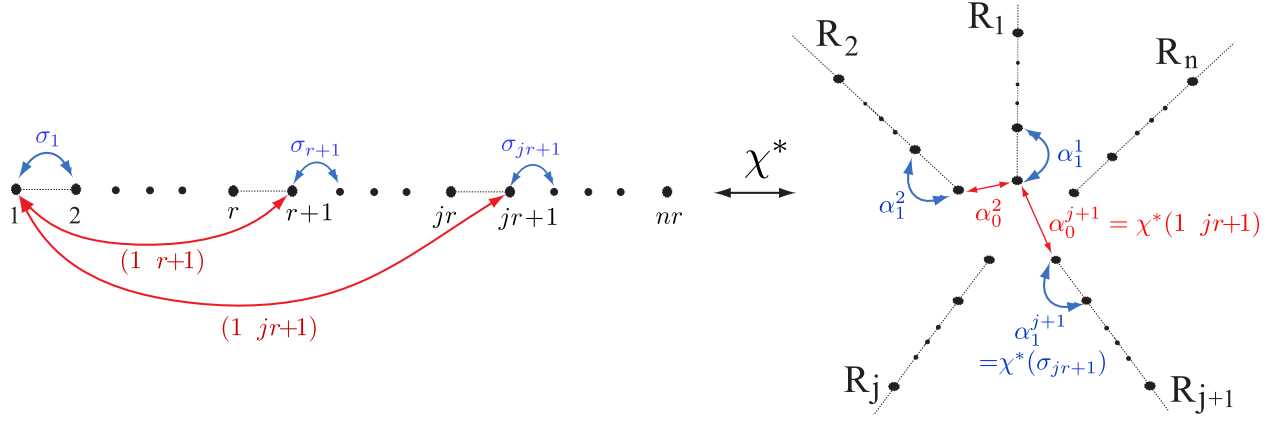


FIGURE 2.4. Correspondence between two generating sets via  $\chi^*$

As in (2.6), define new generators of the second type by

$$(2.8) \quad \alpha_0^{j+1} := \chi^*((\sigma_{jr})^{\overline{w_{jr-1}}}) \text{ for } j = 1, \dots, n-1.$$

Figure 2.4 illustrates how one transforms the generating set of  $S_{nr}$  to a new generating set.

Consider the group  $\Sigma_{n,r}$  presented by

$$(2.9) \quad \Sigma_{n,r} = \langle \alpha_p^i, \alpha_0^{j+1} \mid R \rangle \quad 1 \leq i \leq n, 1 \leq p \leq r-1, \text{ and } 1 \leq j \leq n-1,$$

where  $R'$  consists of

$$\begin{aligned} R'_1: & (\alpha_p^i)^2, (\alpha_0^j)^2, \forall i, p, j \\ R'_2: & [\alpha_p^i, \alpha_q^i], \forall i, |p-q| \geq 2 \\ R'_3: & [\alpha_p^i, \alpha_{p'}^{i'}], \forall p, q, i \neq i' \\ R'_4: & [\alpha_p^i, \alpha_0^{j+1}], \forall j, \alpha_p^i \neq \alpha_1^1, \alpha_1^{j+1} \\ R'_5: & (\alpha_p^i \alpha_{p+1}^i)^3, \forall i, 1 \leq p \leq r-2 \\ R'_6: & (\alpha_1^1 \alpha_0^{j+1})^3, (\alpha_1^{j+1} \alpha_0^j)^3. \end{aligned}$$

Note that  $R$  and  $R'$  share the same idea described in Remark 2.9. Expecting  $R'$  to replace the original relations of  $R$  is reasonable.

**Theorem 2.10.** *With the above definition,  $\Sigma_{n,r} \cong S_{nr}$  for all integers  $n, r \geq 1$ .*

*Proof.* First we show  $\langle\langle R \rangle\rangle \leq \langle\langle R' \rangle\rangle$ . It is clear  $\langle\langle R_1 \rangle\rangle \leq \langle\langle R'_1 \rangle\rangle$ . Basic idea is to use appropriate conjugation relations between  $\sigma_{jr}$  and  $\alpha_0^{j+1}$  given by (2.8).

In order to proceed we need the following identities for  $j = 1, 2, \dots, n-1$

$$(2.10) \quad [\sigma_{jr}, \alpha_p^i] = [\alpha_0^{j+1}, \alpha_p^i]^{w_{jr-1}} \text{ if } (i-1)r + p \geq jr + 2,$$

$$(2.11) \quad [\sigma_{jr}, \alpha_p^i] = [\alpha_0^{j+1}, \alpha_{p+1}^i]^{w_{jr-1}} \text{ if } (i-1)r + p \leq jr - 2,$$

$$(2.12) \quad [\sigma_{jr}, \sigma_{j'r}] = [\alpha_0^{j+1}, \alpha_1^{j'+1}]^{w_{jr-1}} \text{ if } j' < j+1,$$

$$(2.13) \quad (\alpha_1^{j+1} \sigma_{jr})^3 = ((\alpha_1^{j+1} \alpha_0^{j+1})^3)^{w_{jr-1}}$$

$$(2.14) \quad (\alpha_{r-1}^j \sigma_{jr})^3 = ((\alpha_1^1 \alpha_0^{j+1})^3)^{w_{jr-1} w_{jr-2}}$$

$$(2.15) \quad (\alpha_1^1)^{w_{jr-1} w_{jr-2}} = \alpha_{r-1}^j.$$

First we apply an induction argument on  $j$  to verify (2.10), (2.11) and (2.13). The induction assumption for (2.10) together with relators of  $R'_3$  yields  $\alpha_p^i = (\alpha_p^i)^{w_{jr-1}}$  if  $(i-1)r + p \geq jr + 2$ . So (2.10) follows from (2.8). Similarly one can verify (2.11) by using (2.10) together with  $R'_2$  and  $R'_5$ . The identity (2.13) follows from (2.10),  $R'_2$  and  $R'_3$ .

For (2.12), (2.14) and (2.15) we apply simultaneous induction on  $j$  together with (2.10), (2.11) and (2.13). The base case of (2.12) holds trivially. The relation (2.15) is clear when  $j = 1$ , which establishes the base case of (2.14). Suppose (2.12), (2.14) and (2.15) hold for  $j = k$ . Observe that

$$\begin{aligned} & w_{(k+1)r-1} w_{(k+1)r-2} \\ &= (\sigma_1 \cdots \sigma_{kr} \sigma_{kr+1} \cdots \sigma_{(k+1)r-1}) (\sigma_1 \sigma_2 \cdots \sigma_{kr} \sigma_{kr+1} \cdots \sigma_{(k+1)r-2}) \\ &= (\sigma_1 \cdots \sigma_{kr}) (\sigma_1 \sigma_2 \cdots \sigma_{kr-1}) (\sigma_{kr+1} \cdots \sigma_{(k+1)r-1}) (\sigma_{kr} \sigma_{kr+1} \cdots \sigma_{(k+1)r-2}) \quad (2.12), R'_2, R'_3 \\ &= (\sigma_1 \cdots \sigma_{kr-1}) (\sigma_1 \sigma_2 \cdots \sigma_{kr-2} \sigma_{kr} \sigma_{kr-1}) (\sigma_{kr+1} \cdots \sigma_{(k+1)r-1}) (\sigma_{kr} \sigma_{kr+1} \cdots \sigma_{(k+1)r-2}) \quad (2.11), (2.12) \end{aligned}$$

Thus we have

$$(\alpha_1^1)^{w_{(k+1)r-1} w_{(k+1)r-2}} = (\alpha_{r-1}^k)^{(\sigma_{kr} \sigma_{kr-1} \sigma_{kr+1} \cdots \sigma_{(k+1)r-1}) (\sigma_{kr} \sigma_{kr+1} \cdots \sigma_{(k+1)r-2})} \quad (2.14)$$

$$= (\sigma_{kr})^{(\sigma_{kr+1} \cdots \sigma_{(k+1)r-1}) (\sigma_{kr} \sigma_{kr+1} \cdots \sigma_{(k+1)r-2})} \quad (2.13)$$

$$= (\sigma_{kr})^{(\sigma_{kr+1} \sigma_{kr} \sigma_{kr+2} \cdots \sigma_{(k+1)r-1}) (\sigma_{kr+1} \cdots \sigma_{(k+1)r-2})} \quad (2.13)$$

$$= (\alpha_1^{k+1})^{(\sigma_{kr+2} \cdots \sigma_{(k+1)r-1}) (\sigma_{kr+1} \cdots \sigma_{(k+1)r-2})} \quad (2.13)$$

$$= \cdots = \alpha_{r-1}^{k+1} \quad R'_2, R'_5$$

So we verify the statement (2.14) with  $j = k + 1$  which imply (2.13) for  $j = k + 1$ . For (2.12) we need to check  $(\alpha_1^{j'+1})^{w_{(k+1)r-1}} = \sigma_{j'r}$  if  $j' < k + 2$ ;

$$\begin{aligned} (\alpha_1^{j'+1})^{w_{(k+1)r-1}} &= (\alpha_1^{j'+1})^{\sigma_1 \sigma_2 \cdots \sigma_{j'r} \sigma_{j'r+1} \cdots \sigma_{(k+1)r-1}} && \text{definition of } \sigma_{(k+1)r-1} \\ &= (\alpha_1^{j'+1})^{\sigma_{j'r} \sigma_{j'r+1} \cdots \sigma_{(k+1)r-1}} && (2.10), R'_3 \\ &= (\sigma_{(j'+1)r})^{\sigma_{j'r+2} \cdots \sigma_{(k+1)r-1}} && (2.10), (2.13) \\ &= \cdots = \sigma_{(j'+1)r} && (2.10). \end{aligned}$$

Now we see identities (2.10)-(2.14) imply  $\langle\langle R \rangle\rangle \leq \langle\langle R' \rangle\rangle$ . On the other hand, Remark 2.9 guarantees  $\langle\langle R' \rangle\rangle \leq \langle\langle R \rangle\rangle$ . So we have  $\langle\langle R' \rangle\rangle = \langle\langle R \rangle\rangle$

The next step is to apply Tietze transformations to the presentation (2.7) which is isomorphic to Coxeter presentation (2.5) (up to notation for generators). First adjoin extra letters  $\alpha_0^{j+1}$ 's to get the following presentation

$$(2.16) \quad \langle \alpha_p^i, \sigma_{jr}, \alpha_0^{j+1} \mid R \cup R_\beta \rangle$$

where  $R_\beta$  consists of words of the form

$$\overline{(\alpha_0^{j+1})(\sigma_{jr})^{w_{jr-1}}}$$

corresponding to (2.8). Then replace  $R \cup R_\beta$  by  $R' \cup R_\beta$  in (2.16). This is legitimate since two normal closures  $\langle\langle R \rangle\rangle$  and  $\langle\langle R' \rangle\rangle$  are the same. Finally we remove  $\sigma_{jr}$ 's from the generating set and  $R_\beta$  from the relators simultaneously to get the presentation  $\Sigma_{n,r}$ . In all,  $\Sigma_{n,r}$  and  $\Sigma_{nr}$  are isomorphic.  $\square$

Let  $n$  be a positive integer. From Theorem 2.10 we have a sequence of symmetric groups  $\Sigma_{n,r} = \langle A_{n,r} | Q_{n,r} \rangle$  on  $B_{n,r} \subset Y_n$ , where  $A_{n,r}$  and  $Q_{n,r}$  denote the generators and relators in (2.9) respectively. The natural inclusion  $B_{n,r} \hookrightarrow B_{n,r+1}$  induces an inclusion  $i_r : \Sigma_{n,r} \hookrightarrow \Sigma_{n,r+1}$  such that

$$A_{n,r} \hookrightarrow A_{n,r+1} \text{ and } Q_{n,r} \hookrightarrow Q_{n,r+1}$$

for  $r \in \mathbb{N}$ . The direct limit  $\varinjlim_r \Sigma_{n,r}$  with respect to inclusions  $i_r$  is nothing but the infinite symmetric group  $\Sigma_{n,\infty}$  on  $Y_n$  consisting of permutations with finite support.

**Lemma 2.11.** *For each  $n \in \mathbb{N}$ ,  $\Sigma_{n,\infty} \cong \langle A_n | Q_n \rangle$ , where  $A_n = \cup_r A_{n,r}$  and  $Q_n = \cup_r Q_{n,r}$ .*

*Proof.* For each  $k \in \mathbb{N}$  we have a commuting diagram

$$\begin{array}{ccc} \langle \cup_{r=1}^k A_{n,r} | \cup_{r=1}^k Q_{n,r} \rangle & \longrightarrow & \langle A_n | Q_n \rangle \\ \downarrow & & \downarrow \\ \Sigma_{n,k} & \longrightarrow & \Sigma_{n,\infty}. \end{array}$$

The isomorphism on the right follows from the fact that the left map is an isomorphism and that the two horizontal maps are inclusions.  $\square$

The previous Lemma implies that one can get a presentation for  $\Sigma_{n,\infty}$  by allowing the subscript  $p$  to be any positive integer in (2.9). We keep the same notation  $R'_1$ - $R'_6$  to indicate relators of the same type in  $Q_n$ .

Recall elements  $g_1, \dots, g_{n-1}$  of  $\mathcal{H}_n$  defined in (2.1), and it is easy to see that

$$[g_i, g_j] = \alpha_1^1$$

for  $i \neq j$ . Now consider the action of  $g_1, \dots, g_{n-1}$  on  $A_n$ .

$$(\alpha_p^1)^{g_k^{-m}} = \alpha_{p+m}^1, \quad (\alpha_1^1)^{g_k} = \alpha_0^k, \quad \text{for } p, m \in \mathbb{N}, 1 \leq k \leq n-1$$

$$(\alpha_0^k)^{g_k} = \alpha_1^{k+1}, \quad \text{for } 1 \leq k \leq n-1$$

$$(\alpha_0^k)^{g_{k'}^{-1}} = (\alpha_0^k)^{\alpha_1^1}, \quad (\alpha_0^k)^{g_{k'}} = (\alpha_0^k)^{\alpha_0^{k'}}, \quad \text{for } 1 \leq k \neq k' \leq n-1$$

$$(\alpha_p^i)^{g_{i-1}^m} = (\alpha_{p+m}^i), \quad (\alpha_p^i)^{g_j^{-m}} = (\alpha_p^i)^{g_j^m} = (\alpha_p^i), \quad \text{for } p, m \in \mathbb{N}, 2 \leq i \leq n \text{ and } j \neq i-1$$

Let  $Q'_n$  be the relators corresponding to the above action. For  $n \geq 3$ , consider the group  $H_n$  defined by

$$(2.17) \quad H_n = \langle g_1, \dots, g_{n-1}, A_n | Q_n, Q'_n, (\alpha_1^1)^{-1}[g_i, g_j] \rangle, \quad 1 \leq i < j \leq n-1$$

**Lemma 2.12.** *We have a short exact sequence for each  $n$ ;*

$$(2.18) \quad 1 \rightarrow \langle \mathcal{A}_n \rangle \rightarrow H_n \rightarrow \mathbb{Z}^{n-1} \rightarrow 1.$$

*Proof.* Note that the relations  $Q'_n$  ensure that  $\langle A_n \rangle$  is a normal subgroup of  $H_n$ . Clearly the set of relations  $Q_n$  becomes trivial in the quotient  $H_n/\langle A_n \rangle$ . Likewise, the relations in  $Q'_n$  all become trivial in the quotient  $H_n/\langle A_n \rangle$ . The extra relator  $(\alpha_1^1)^{-1}[g_i, g_j]$  implies that  $g_i$  and  $g_j$  commute in  $H_n/\langle A_n \rangle$  ( $i \neq j$ ), so the quotient is the free abelian group generated by  $g_1, \dots, g_{n-1}$ .  $\square$

**Lemma 2.13.** *For  $n \geq 3$ ,  $H_n \cong \mathcal{H}_n$ .*

*Proof.* As in Johnson's argument for  $n = 3$ , there exists a homomorphism  $\psi : H_n \rightarrow \mathcal{H}_n$  defined by

$$\psi(g_i) = g_i \text{ and } \psi(\alpha) = \alpha.$$

So we have a commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \langle \mathcal{A}_n \rangle & \longrightarrow & H_n & \longrightarrow & H_n/\langle \mathcal{A}_n \rangle & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & [\mathcal{H}_n, \mathcal{H}_n] & \longrightarrow & \mathcal{H}_n & \longrightarrow & \mathbb{Z}^{n-1} & \longrightarrow & 1 \end{array}$$

The existence of the top row and the isomorphism  $H_n/\langle A_n \rangle \simeq \mathbb{Z}^{n-1}$  are the result of Lemma 2.12. The bottom row is given by Corollary 2.4, and the isomorphism  $\langle A_n \rangle \simeq [H_n, H_n]$  follows from Lemma 2.3 and Lemma 2.11. As the outer four vertical maps are isomorphisms, so is the inner one by the Five lemma.  $\square$

**Theorem 2.14.** *For  $n \geq 3$ ,  $H_n$  has a finite presentation*

$$(2.19) \quad H_n \cong \langle g_1, \dots, g_{n-1}, \alpha \mid P_n \rangle$$

where  $P_n$  consists of

$$\begin{aligned} r'_1 & : \alpha^2 = 1, \\ r'_2 & : (\alpha \alpha^{g_1})^3 = 1, \\ r'_3 & : \alpha \sim \alpha^{\bar{g}_1}, \\ r'_4 & : [g_i, g_j] = \alpha, \text{ for } 1 \leq i < j \leq n-1, \\ r'_5 & : \alpha^{\bar{g}_i} = \alpha^{\bar{g}_j}, \text{ for } 1 \leq i < j \leq n-1 \end{aligned}$$

( $\sim$  denotes commutation).

*Proof.*  $P_n$  already contains  $(\alpha_1^1)^{-1}[g_i, g_j]$  for  $1 \leq i < j \leq n-1$ . We show  $\langle\langle P_n \rangle\rangle$  contains  $\langle\langle \mathcal{Q}'_n \cup \mathcal{Q}_n \rangle\rangle$  for all  $n \geq 3$  by induction on  $n$ . The base case is established by Theorem 2.8. Assume  $H_n$  has the presentation as in 2.19 for  $n \geq 3$ . Obviously  $g_1, \dots, g_n$  and  $\alpha$  generate  $\mathcal{H}_{n+1}$ . Consider the natural inclusion of  $\iota : Y_n \rightarrow Y_{n+1}$  such that  $\iota(R_i)$  is the  $i^{\text{th}}$  ray of  $Y_{n+1}$ . This induces an embedding  $\iota : H_n \rightarrow H_{n+1}$ , and  $\mathcal{Q}'_{n+1}$  contains relations corresponding the action of  $g_n$  on  $\alpha_0^n$  and  $\alpha_p^{n+1}$ 's:

$$(2.20) \quad (\alpha_0^n)^{g_n} = \alpha_1^{n+1}, \quad (\alpha_0^n)^{\bar{g}_n} = \alpha_1^1, \quad (\alpha_p^{n+1})^{g_n} = \alpha_{p+1}^{n+1}$$

for  $p \in \mathbb{N}$ . This implies  $\langle\langle \iota(\mathcal{Q}_n) \rangle\rangle = \langle\langle \mathcal{Q}_{n+1} \rangle\rangle$  in  $H_{n+1}$ . So the normal closure of  $P_{n+1}$  contains  $\langle\langle \mathcal{Q}_{n+1} \rangle\rangle$  by the induction assumption.

We must examine the action of  $g_1, \dots, g_n$  on  $\mathcal{A}_n$ 's:

$$(2.21) \quad (\alpha_p^1)^{\bar{g}_n} = (\alpha_p^1)^{\bar{g}_i}, \quad k \in \mathbb{N}, \quad 1 \leq i \leq n-1,$$

$$(2.22) \quad (\alpha_p^i)^{g_n} = \alpha_p^i, \quad p \in \mathbb{N}, \quad 2 \leq i \leq n-1,$$

$$(2.23) \quad (\alpha_0^i)^{g_n} = (\alpha_0^i)^{\alpha_0^n}, (\alpha_0^i)^{\bar{g}_n} = (\alpha_0^i)^{\alpha_0^1}, 1 \leq i \leq n-1,$$

$$(2.24) \quad (\alpha_0^n)^{\bar{g}_i} = (\alpha_0^n)^{\alpha_0^1}, (\alpha_0^n)^{g_i} = (\alpha_0^{n+1})^{\alpha_0^i}, 1 \leq i \leq n-1,$$

$$(2.25) \quad (\alpha_p^{n+1})^{g_i} = \alpha_p^{n+1}, k \in \mathbb{N}, 1 \leq i \leq n-1,$$

together with (2.20) and  $\iota(Q'_n)$ . From Lemma 2.15 we see that  $P_{n+1}$  implies

$$(2.26) \quad \alpha \sim (\bar{g}_n)^{\bar{g}_i^2}, \alpha \sim (\bar{g}_i)^{\bar{g}_n^2}, 1 \leq i \leq n-1,$$

$$(2.27) \quad \alpha^{\bar{g}_i^k} = \alpha^{\bar{g}_n^k}, \alpha \sim \alpha^{\bar{g}_i^{(k+1)}}, k \in \mathbb{N}, 1 \leq i \leq n-1$$

We want to establish identities (2.21)-(2.25) from (2.26), (2.27) and  $P_{n+1}$ . The identity (2.21) is a part of (2.27). Observe that, by using (2.20), one can reduce (2.22) and (2.25) respectively, as follows

$$(2.28) \quad \alpha^{g_i^{k+1}} \sim g_n, \alpha^{g_n^{k+1}} \sim g_i, 2 \leq i \leq n-1, k \in \mathbb{N},$$

One can verify (2.28) by inductions based on (2.26) as follows. The identity (2.26) provides the base cases. Assume  $\alpha^{g_i^{k+1}} \sim g_n$  for  $k \geq 1$ . Then  $r_1$  and  $r_2$  of  $P_{n+1}$  imply

$$\alpha^{g_i^{k+2}} \sim \bar{g}_i g_n g_i = \bar{g}_i \alpha g_i g_n = \alpha^{g_i} g_n$$

On the other hand,  $\alpha^{g_i^{k+2}} \sim \alpha^{g_i}$  by (2.27). Thus  $\alpha^{g_i^{k+2}} \sim g_n$ . An analogous argument verifies the second identity of (2.28).

Note that (2.23) and (2.24) are equivalent to

$$\alpha^{g_i} \sim \bar{g}_n \alpha, \alpha^{g_n} \sim \alpha g_i, 1 \leq i \leq n-1,$$

which are immediate consequence of (2.26).

In all,  $\langle\langle P_{n+1} \rangle\rangle$  also contains  $\langle\langle Q'_{n+1} \rangle\rangle$ . So  $P_{n+1}$  is enough for relators in  $H_{n+1}$ , and hence  $H_n$  has the presentation (2.19) for  $n \geq 3$ .  $\square$

**Lemma 2.15.**  $P_n$  implies the following identities

$$(2.29) \quad \alpha \sim (\bar{g}_{n-1})^{\bar{g}_i^2}, \alpha \sim (\bar{g}_i)^{\bar{g}_{n-1}^2}, 1 \leq i \leq n-2,$$

$$(2.30) \quad \alpha^{\bar{g}_i^k} = \alpha^{\bar{g}_{n-1}^k}, \alpha \sim \alpha^{\bar{g}_i^{(k+1)}}, k \in \mathbb{N}, 1 \leq i \leq n-2$$

*Proof.* From  $r_1, r_2$  and  $r_5$  we have

$$\alpha \alpha^{\bar{g}_i} \alpha = \alpha^{\bar{g}_i} \alpha \alpha^{\bar{g}_i} \Rightarrow \alpha^{\bar{g}_i} \alpha = \alpha^{g_i \alpha \bar{g}_i} \Rightarrow \alpha^{g_i} \alpha = \alpha^{\bar{g}_i \alpha g_i} = \alpha^{g_{n-1}^{-1} \alpha g_i} = \alpha^{g_i \bar{g}_{n-1}}.$$

Thus,

$$\alpha^{g_i} \sim \bar{g}_{n-1} \alpha = g_i \bar{g}_{n-1} \bar{g}_i \Rightarrow \alpha \sim (\bar{g}_{n-1})^{\bar{g}_i^2}.$$

One obtains  $\alpha \sim (\bar{g}_i)^{\bar{g}_{n-1}^2}$  by using an analogous argument. Next, to establish (2.30), we use simultaneous induction on  $k$  together with (2.29). Observe that  $r_4$  and  $r_5$  provide the base case  $k = 1$  and that  $\alpha^{\bar{g}_i^k} = \alpha^{\bar{g}_{n-1}^k}$  holds trivially when  $k = 0$ . Now assume

$$(2.31) \quad \alpha^{\bar{g}_i^{(k-2)}} = \alpha^{\bar{g}_{n-1}^{(k-2)}}, \alpha^{\bar{g}_i^{(k-1)}} = \alpha^{\bar{g}_{n-1}^{(k-1)}}, \alpha \sim \alpha^{\bar{g}_i^k}$$

for  $k \geq 2$ . From  $r_1$  and (2.30),

$$\alpha^{\bar{g}_i^k} = \alpha^{\bar{g}_i^k} \alpha = \alpha^{\bar{g}_{n-1}^{(k-1)} \bar{g}_i} \alpha = \alpha^{\bar{g}_{n-1}^{(k-2)} \bar{g}_i \bar{g}_{n-1}} = \alpha^{\bar{g}_i^{(k-1)} \bar{g}_i \bar{g}_{n-1}} = \alpha^{\bar{g}_{n-1}^k}.$$

For the second assertion, note that  $r_4$  and the first identity of (2.29) imply that  $\alpha^{g_i^2}$  commute with both  $\alpha$  and  $g_{n-1}^{-1}$ . So we have

$$\alpha^{g_i^2} \sim \alpha^{\bar{g}_{n-1}^{(k-1)}} = \alpha^{\bar{g}_i^{(k-1)}},$$

or equivalently,

$$\alpha \sim \alpha^{\bar{g}_i^{(k+1)}}.$$

□

As a consequence of Lemma 2.13 and Theorem 2.14 we have

**Theorem C.** *For  $n \geq 3$ ,  $\mathcal{H}_n$  has a finite presentation*

$$(2.32) \quad \mathcal{H}_n \cong \langle g_1, \dots, g_{n-1}, \alpha \mid P_n \rangle$$

where  $P_n$  is the same as in the presentation (2.19) of Theorem 2.14.

#### 2.4. Further properties of $\mathcal{H}_n$ .

**Definition 2.16 (Amenable and a-T-menable groups).** A (discrete) group  $G$  is called *amenable* if it has a *Følner* sequence, i.e., there exists a sequence  $\{F_i\}_{i \in \mathbb{N}}$  of finite subsets of  $G$  such that

$$\frac{|g \cdot F_i \triangle F_i|}{|F_i|}$$

tends to 0 for each  $g \in G$ .

A group  $G$  is called *a-T-menable* if there is a proper continuous affine action of  $G$  on a Hilbert space.

Recall the following known fact about amenable groups ([16]).

**Theorem 2.17.** *The class of amenable group contains abelian groups, and locally finite groups. An extension of an amenable group by another amenable group is again amenable.*

The above facts together with the short exact sequence (2.3) implies amenability of  $\mathcal{H}_n$ .

$$1 \rightarrow \Sigma_{n,\infty} \rightarrow \mathcal{H}_n \xrightarrow{\varphi} \mathbb{Z}^{n-1} \rightarrow 1,$$

**Theorem 2.18.**  *$\mathcal{H}_n$  is amenable for all  $n$ .*

The class of amenable groups contains the class of a-T-menable groups.

**Corollary 2.19.**  *$\mathcal{H}_n$  is a-T-amenable for all  $n$ . As a consequence, Houghton's groups satisfy the Baum-Connes conjecture.*

### 3. A CUBING FOR $\mathcal{H}_n$

**3.1. Definition of the cubing  $X_n$ .** In [6], Ken Brown constructed an infinite dimensional cell complex on which  $\mathcal{H}_n$  acts by taking the geometric realization of a poset, which is defined using factorizations in a monoid  $\mathcal{M}_n$  containing  $\mathcal{H}_n$ . In this section we modify his idea to construct an  $n$ -dimensional cubical complex  $X_n$  on which  $\mathcal{H}_n$  acts. It turns out that  $X_n$  is a cubing for all  $n \in \mathbb{N}$ . We discuss this in detail in Section 3.3.

Two monoids  $\mathcal{M}_n$  and  $\mathcal{T}_n$  are important for the construction. Fix a positive integer  $n \in \mathbb{N}$ . Let  $\mathcal{M}_n$  be the monoid of injective maps  $Y_n \rightarrow Y_n$  which behave as eventual translations, i.e., each  $\alpha \in \mathcal{M}_n$  satisfies

- ★ There is an  $n$ -tuple  $(m_1, \dots, m_n) \in \mathbb{Z}^n$  and a finite set  $K \subset Y_n$  such that  $\alpha(k, p) = (k, p + m_k)$  for all  $(k, p) \in Y_n - K$ .

The group homomorphism  $\varphi : \mathcal{H}_n \rightarrow \mathbb{Z}^n$  naturally extends to a monoid homomorphism  $\varphi : \mathcal{M}_n \rightarrow \mathbb{Z}^n$  and  $\varphi(\alpha) = (m_1, \dots, m_n)$ . A monoid homomorphism  $h : \mathcal{M}_n \rightarrow \mathbb{Z}_{\geq 0}$  is defined by  $h(\alpha) = \sum m_i$ . This map  $h$  is called the *height function* on  $\mathcal{M}_n$ . For convenience let  $S(\alpha)$  denote the discrete set  $Y_n - (Y_n)\alpha$ . One can readily check that  $h(\alpha) = |S(\alpha)|$  and that  $\mathcal{H}_n = h^{-1}(0)$ .

Consider elements  $t_1, \dots, t_n$  of  $\mathcal{M}_n$  where  $t_i$  is the *translation* by 1 on the  $i^{\text{th}}$  ray, i.e.,

$$(j, p)t_i = \begin{cases} (j, p + 1) & \text{if } j = i \\ (j, p) & \text{if } j \neq i. \end{cases}$$

for all  $p \in \mathbb{N}$ . Let  $\mathcal{T}_n \subset \mathcal{M}_n$  be the commutative submonoid generated by  $t_1, \dots, t_n$ .

Figure 3.1 illustrates some examples of  $\mathcal{M}_n$ , as before, where points which do not involve arrows are meant to be fixed, points of each finite set  $K$  are indicated by circles. The left most one is a generator  $t_i$  of  $\mathcal{T}_n$  which behaves as the translation on  $R_i$  by 1 and fixes  $Y_n - R_i$  pointwise. The next example shows the commutativity of  $\mathcal{T}_n$ ;  $t_i t_j = t_j t_i$ . In the third figure,  $\alpha = [g_1, g_2]$  as before. The last element  $g$  is rather generic one with  $h(g) = 1$ .

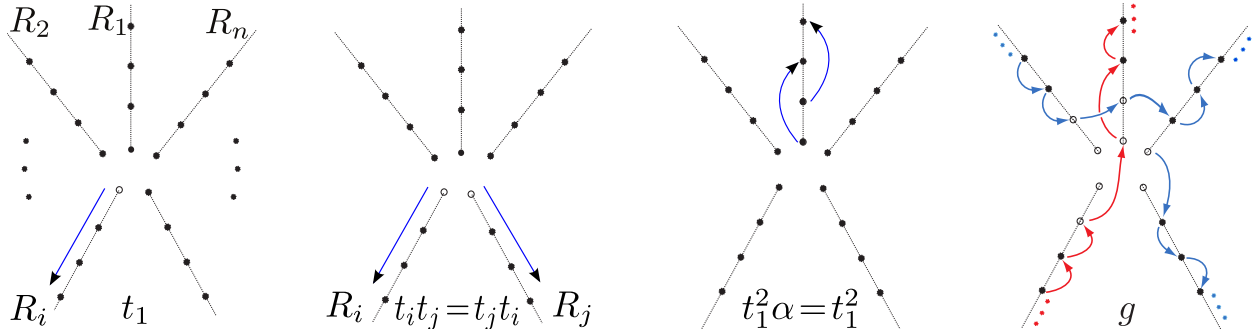


FIGURE 3.1. Some elements of  $\mathcal{M}_n$

One crucial feature of  $\mathcal{T}_n$  is that it is commutative.  $\mathcal{T}_n$  acts on  $\mathcal{M}_n$  by left multiplication (pre-composition). This action induces a relation  $\geq$  on  $\mathcal{M}_n$ , i.e.,

$$(3.1) \quad \alpha_1 \geq \alpha_2 \text{ if } \alpha_1 = t\alpha_2 \text{ for some } t \in \mathcal{T}_n.$$

for all  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}_n$

**Proposition 3.1.** *The relation  $\geq$  on  $\mathcal{M}_n$  is a partial order, i.e., it satisfies for all  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}_n$ ,*

- $\alpha_1 \geq \alpha_1$ ,
- if  $\alpha_1 \geq \alpha_2$  and  $\alpha_2 \geq \alpha_1$  then  $\alpha_1 = \alpha_2$ ,
- if  $\alpha_1 \geq \alpha_2$  and  $\alpha_2 \geq \alpha_3$  then  $\alpha_1 \geq \alpha_3$ .

*Proof.* Reflexivity can be verified by using  $t = 1 \in \mathcal{T}_n$  in equation 3.1. The condition  $\alpha_1 \geq \alpha_2$  and  $\alpha_2 \geq \alpha_1$  implies that  $h(\alpha_1) = h(\alpha_2)$  and so  $t = 1$  is the only candidate for  $t$  in those inequalities. So we have antisymmetry. One can check transitivity by taking an obvious composition of  $t$ 's in the two inequalities.  $\square$

Being a submonoid,  $\mathcal{H}_n$  acts on  $\mathcal{M}_n$ . In this paper, we focus on the action of  $\mathcal{H}_n$  on  $\mathcal{M}_n$  by right multiplication (post-composition).

**Remark 3.2.** The two actions of  $\mathcal{H}_n$  and  $\mathcal{T}_n$  on  $\mathcal{M}_n$  commute.

A chain  $\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_k$  corresponds to a  $k$ -simplex in the geometric realization  $\|\mathcal{M}_n\|$ . The length of chains is not bounded and hence the dimension of  $\|\mathcal{M}_n\|$  is infinite. To show the finiteness properties of  $\mathcal{H}_n$  (Corollary B), Brown studied the action of  $\mathcal{H}_n$  on  $\|\mathcal{M}_n\|$  equipped with a filtration satisfying the condition of Theorem 3.51.

Instead, for the construction of desired complex  $X_n$ , we want to start with Cayley graph of  $\mathcal{M}_n$  with respect to the generating set  $\{t_1, \dots, t_n\}$  for  $\mathcal{T}_n$ . Recall the *Cayley graph*  $\mathcal{C}_{\mathcal{A}}(\Gamma)$  of a group  $\Gamma$  with respect to a generating set  $\mathcal{A}$  is the metric graph whose vertices are in 1-1 correspondence with the elements of  $\Gamma$  and which has an edge (labelled by  $a$ ) of length 1 joining  $\gamma$  to  $\gamma a$  for each  $\gamma \in \Gamma$  and  $a \in \mathcal{A}$ .

**Definition 3.3 (Cayley graph  $\mathcal{C}_n$ ).** Associated to the monoid  $\mathcal{M}_n$ , a directed graph  $\mathcal{C}_n$  is defined by

- The vertex set  $\mathcal{C}_n^{(0)}$  is in 1-1 correspondence with the elements of  $\mathcal{M}_n$ . By abusing notation, let  $\alpha$  denote the vertex of  $\mathcal{C}_n$  corresponding the same element of  $\mathcal{M}_n$ .
- A vertex  $\alpha$  is joined to another vertex  $\beta$  by an edge  $e$  (of length 1) labelled by  $t_i$  if  $\beta = t_i \alpha$  for some  $t_i$ ; we assume all edges are directed, i.e., directed from  $\alpha$  to  $\beta$ .

The monoid homomorphism  $h : \mathcal{M}_n \rightarrow \mathbb{Z}_{\geq 0}$  can be extended linearly to a map  $h : \mathcal{C}_n \rightarrow \mathbb{R}_{\geq 0}$ . Note that  $h(\beta) = h(\alpha) + 1$  if  $\beta = t_i \alpha$  for some  $t_i$ . In this situation, we say that the edge  $e_{t_i}$  is directed “upward” with respect to the height function  $h$ .

**Remark 3.4.** For a vertex  $\beta$  and a fixed  $i$  ( $1 \leq i \leq n$ ), there is unique edge labelled by  $t_i$  whose initial vertex is  $\beta$ ; namely, the edge which ends in the uniquely determined vertex  $t_i \beta$ .

### Least upper bound in $\mathcal{M}_n$ .

**Lemma 3.5.** For  $\alpha_1, \alpha_2 \in \mathcal{M}_n$ , there exists a unique element  $\beta$  such that

- $\beta \geq \alpha_1$  and  $\beta \geq \alpha_2$ , and
- if  $\beta' \geq \alpha_1$  and  $\beta' \geq \alpha_2$ , then  $\beta' \geq \beta$ .

*Proof.* Suppose  $\varphi(\alpha_1) = (m_1, \dots, m_n)$  and  $\varphi(\alpha_2) = (m'_1, \dots, m'_n)$ . Consider the restrictions  $\alpha_1|_{R_i}$  and  $\alpha_2|_{R_i}$  for  $i = 1, \dots, n$ . One can find smallest non-negative integers  $k_i, k'_i$  so that

$$(3.2) \quad t_i^{k_i} \alpha_1|_{R_i} = t_i^{k'_i} \alpha_2|_{R_i}$$

for each  $i$  as follows. If  $m_i > m'_i$  then there exists smallest  $p_0 \in \mathbb{N} \cup \{0\}$  such that

$$(i, p) \alpha_1 = (i, p) t_i^{m_i - m'_i} \alpha_2$$

for all  $p > p_0$ . The existence of  $p_0$  is obvious. On  $R_i$ ,  $\alpha_1$  and  $t_i^{m_i - m'_i} \alpha_2$  agree outside a finite set  $L$  (this set  $L$  can be smaller than the finite set in the definition of  $\mathcal{M}_n$ ). The integer  $k_0$  is determined by the point in  $L \cap R_i$  with the largest distance from the origin (if the intersection is trivial then  $p_0 = 0$ ). Set  $k_i = p_0$  and  $k'_i = m_i - m'_i + p_0$ . The integers  $k_i$  and  $k'_i$  are desired powers for  $t_i$  satisfying (3.2). One can find appropriate  $k_i$  and  $k'_i$  in a similar way in case  $m_i < m'_i$ . If  $m_i = m'_i$  then set  $k_i = k'_i = p_0$ . Apply this process to get  $k_i$ 's and  $k'_i$ 's for all  $i = 1, \dots, n$ .



Now set  $\beta = t_1^{k_1} \cdots t_n^{k_n} \alpha_1 = t_1^{k'_1} \cdots t_n^{k'_n} \alpha_2$ . The first condition clearly holds. The definition of  $\geq$  together with the minimality of the  $k_i$  and  $k'_i$  ensure that if  $\beta' \geq \alpha$  and  $\beta' \geq \alpha'$ , then  $\beta' \geq \beta$ , and hence condition two holds. Finally, the uniqueness of  $\beta$  follows from the second condition together with the antisymmetry of  $\geq$ . (Proposition 3.1).  $\square$

**Definition 3.6 (Least upper bound).** An element  $\beta$  is called an *upper bound* of  $\alpha_1$  and  $\alpha_2$  if it satisfies the first condition of Lemma 3.5. If  $\beta$  is the unique element satisfying both conditions of Lemma 3.5, then it is called the *least upper bound* (simply *lub*) of  $\alpha_1$  and  $\alpha_2$  and is denoted by  $\alpha_1 \wedge \alpha_2$ .

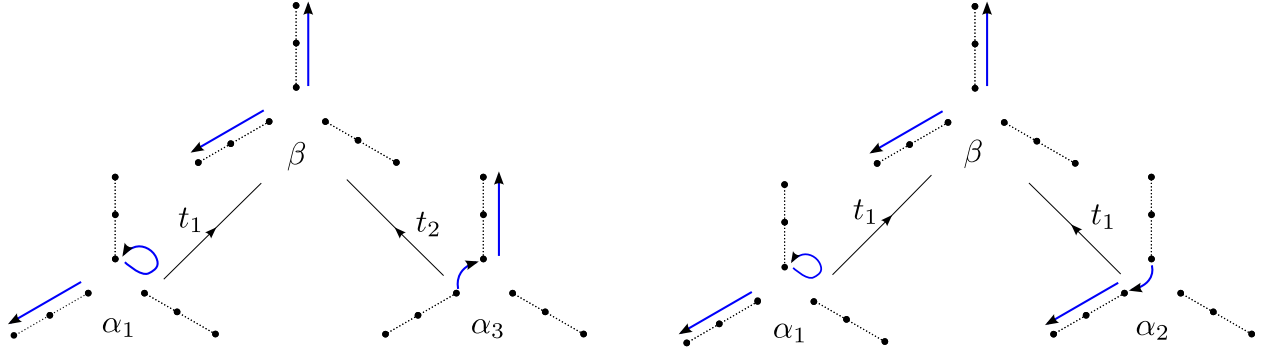


FIGURE 3.2.  $\beta$  is an upper bound of  $\alpha_1, \alpha_2$  and  $\alpha_3$

Figure 3.2 shows some elements of  $\mathcal{M}_n$  with their upper bound;  $\beta$  is an upper bound of  $\alpha_1, \alpha_2$  and  $\alpha_3$ . Note that, in the right figure, edges directed into the vertex  $\beta$  are not necessarily labelled by distinct generators of  $\mathcal{T}_n$ . The reader should contrast this with the situation of edges directed away from a vertex as described in Remark 3.4.

The following proposition implies that the notion of *lub* for a finite collection of  $\mathcal{M}_n$  is well-defined.

**Proposition 3.7.** Suppose  $\{\alpha_1, \dots, \alpha_\ell\} \subset \mathcal{M}_n$ . There exists *lub* of those finite elements, i.e. there exists a unique element  $\beta$  satisfying

- $\beta \geq \alpha_j, j = 1, \dots, \ell$ ,
- if  $\beta' \geq \alpha_j$  for all  $j$  then  $\beta' \geq \beta$ .

*Proof.* We extend the idea in the proof of Lemma 3.5 to find  $\beta$  with desired properties. Fix  $i$  ( $1 \leq i \leq n$ ). Suppose the translation lengths of  $\alpha_j$  ‘at infinity’ is  $m_{i,j}$  (i.e.,  $\alpha_j$  has image  $m_{i,j}$  under the composition  $\mathcal{M}_n \xrightarrow{\varphi} \mathbb{Z}^n \xrightarrow{\pi_i} \mathbb{Z}$  where  $\pi_i$  is the projection to the  $i^{th}$  component). We want to find smallest non-negative integers  $k_{i,1}, \dots, k_{i,\ell}$  so that

$$t_i^{k_{i,j}} \alpha_j|_{R_i} = t_i^{k_{i,j'}} \alpha_{j'}|_{R_i}$$

for all  $j, j' \in \{1, \dots, \ell\}$ . Let  $M_i = \max\{m_{i,1}, \dots, m_{i,\ell}\}$ . Observe that, for  $j = 1, \dots, \ell$ , elements  $t_i^{M_i - m_{i,j}} \alpha_j$  act as a translation on  $R_i$  by  $M_i$  outside a finite set  $K$ . They agree on  $R_i$  outside a finite set  $L$  (as in the proof Lemma 3.5,  $L$  can be smaller than  $K$ ). So there exists smallest  $p_0 \in \mathbb{N} \cup \{0\}$  such that

$$(i, p) \alpha_j = (i, p) \alpha_{j'}$$

for all  $p > p_0$  and  $j, j' \in \{1, \dots, \ell\}$ . Now set  $k_{i,j} = M_i - m_j + p_0$  for  $j = 1, \dots, \ell$ . Apply this process to find  $k_{i,j}$  for all  $i = 1, \dots, n$ . As before, we define

$$(3.3) \quad \beta = t_1^{k_{1,j}} t_2^{k_{2,j}} \dots t_n^{k_{n,j}} \alpha_j$$

for some  $j \in \{1, \dots, \ell\}$ . Note it is well-defined because  $t_1^{k_{1,j}} \dots t_n^{k_{n,j}} \alpha_j = t_1^{k_{1,j'}} \dots t_n^{k_{n,j'}} \alpha_{j'}$  for any  $j' \in \{1, \dots, \ell\}$ . The rest of the argument is analogous to the proof of Lemma 3.5. By the definition (3.3) of  $\beta$  it satisfies the first condition. The second condition is also satisfied by the minimality of powers  $k_{i,j}$ . Finally, the uniqueness follows from the antisymmetry of  $\geq$ .  $\square$

**Remark 3.8.** The operator  $\text{lub}$  preserves the inclusion between finite sets of  $\mathcal{M}_n$ , i.e.,

$$\text{lub}(C_1) \leq \text{lub}(C_2)$$

if  $C_1 \subset C_2 \subset \mathcal{M}_n$ .

### Cubical structure of $\mathcal{C}_n$ and the definition of $X_n$ .

Commutativity of  $\mathcal{T}_n$  plays an important role in the construction of  $X_n$ ; any permutation in the product  $t_1 t_2 \dots t_k$ ,  $k \leq n$ , represents the same element in  $\mathcal{T}_n$ . Yet each variation in the expression  $t_1 t_2 \dots t_k$  determines a path from  $\alpha_1$  to  $t_1 t_2 \dots t_k \alpha_1$ . These  $k!$  many paths form the 1-skeleton of a  $k$ -cube. Figure 3.3 illustrates  $3!$  paths from  $\beta$  to  $t_i t_j t_k \beta$  which form 1-skeleton of a 3-cube having  $\beta$  and  $t_i t_j t_k \beta$  as its bottom and top vertex respectively.

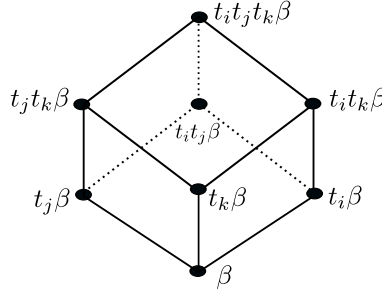


FIGURE 3.3. Three distinct generators  $t_i, t_j, t_k$  determine a 3-cube in  $X_n$ .

**Definition 3.9 (Cubical complex  $X_n$ ).** For each  $n \in \mathbb{N}$   $X_n$  is defined inductively as follows

- (1)  $X_n^{(1)} := \mathcal{C}_n$ ,
- (2) for each  $k \geq 2$ ,  $X_n^{(k)}$  is obtained from  $X_n^{(k-1)}$  by attaching a  $k$ -cube along every copy of the boundary of a  $k$ -cube in  $X_n^{(k-1)}$ .

The height function  $h : \mathcal{C}_n \rightarrow \mathbb{R}_{\geq 0}$  extends linearly to a Morse function  $h : X_n \rightarrow \mathbb{R}_{\geq 0}$ . In our study, a Morse function  $h$  is roughly a height function which restricts to each  $k$ -cube to give the standard height function. (See Definition 3.43 for a Morse function) Let  $\square^k$  denote the standard  $k$ -cube, i.e.,  $\square^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid 0 \leq x_i \leq 1\}$ . The standard height function  $\bar{h} : \square^k \rightarrow \mathbb{R}$  is defined by  $\bar{h}(x_1, \dots, x_k) = \sum x_i$ . Our Morse function  $h$  can be described as follows. Let  $\sigma_j^k \subset X_n$  be a  $k$ -cube and let  $\varphi_j : \square^k \rightarrow \sigma_j^k \subset X_n$  denote the attaching map (isometric embedding) for  $\sigma_j^k$  used in the construction of  $X_n$ . The image of

$\sigma_j^k = \varphi_j(\square^k)$  under  $h$  is the translation of  $\bar{h}(\square^k)$  by  $h\varphi_j(0)$ . In other words, the following diagram commutes.

$$\begin{array}{ccc}
X_n & \xrightarrow{h} & \mathbb{R} \\
\cup & & \uparrow \\
\sigma_j^k & \xrightarrow{\quad} & \mathbb{R} \\
\uparrow \varphi_j & & \uparrow + (h \circ \varphi_j(0)) \\
\square^k & \xrightarrow{\bar{h}} & \mathbb{R}^k_0
\end{array}$$

FIGURE 3.4. A Morse function  $h$  restricted to a cube is the standard height function up to a translation.

Two actions of  $\mathcal{H}_n$  and  $\mathcal{T}_n$  on  $\mathcal{M}_n$  extend to actions on  $X_n$ . Note that cell(cube) structure of  $X_n$  is completely determined by  $\mathcal{T}_n$  whose action commutes with the action of  $\mathcal{H}_n$ . So the action of  $\mathcal{H}_n$  on  $X_n$  is cellular, i.e., it preserves cell structure. If a cube  $\sigma$  is spanned by a collection of vertices  $\{\alpha_1, \dots, \alpha_m\}$ , then the cube  $\sigma \cdot g$  is the same dimensional cube spanned by  $\{\alpha_1 g, \dots, \alpha_m g\}$ .

Intuitively the action of  $\mathcal{H}_n$  can be considered as a ‘horizontal’ action. Each  $g \in \mathcal{H}_n$  preserves the height. By contrast, the action of  $\mathcal{T}_n$  is ‘vertical’. Each non trivial element of  $\mathcal{T}_n$  increases the height under the action. We discuss these actions on  $X_n$  further in section 3.6.

**Lemma 3.10.** *The graph  $\mathcal{C}_n$  is simplicial and connected for all  $n \in \mathbb{N}$ .*

*Proof.* Connectedness is an immediate consequence of the existence of an upper bound for any pair of vertices in  $\mathcal{M}_n$  (Lemma 3.5). Suppose  $\alpha_1$  and  $\alpha_2$  are vertices of  $\mathcal{C}_n$ . Let  $\beta$  be a vertex with

$$\beta = \tau_1 \alpha_1 \text{ and } \beta = \tau_2 \alpha_2$$

for some  $\tau_1, \tau_2 \in \mathcal{T}_n$ . Observe that  $\tau_1$  and  $\tau_2$  define paths  $p_1$  and  $p_2$  where  $p_i$  joins  $\alpha_i$  to  $\beta$ ,  $i = 1, 2$ . So  $\alpha_1$  and  $\alpha_2$  are connected by the concatenation of the two paths.

If an edge connects two vertices  $\beta_1$  and  $\beta_2$  then their heights differ by 1. So no edge in  $\mathcal{C}_n$  starts and ends at the same vertex. Finally we want to show there is no bigon in  $\mathcal{C}_n$ . Suppose two distinct edges share two vertices  $\beta_1$  and  $\beta_2$ . Again, those two vertices can not have the same height and so we may assume  $\beta_1 \geq \beta_2$ . Our claim is that those two edges are labelled by one generator of  $\mathcal{T}_n$ . If two edges were labelled by two distinct generators  $t_i$  and  $t_j$ , then we would have  $\beta_1 = t_i \beta_2$  and  $\beta_1 = t_j \beta_2$ . This contradicts the fact that  $\beta_1$  is injective because

$$(i, 1)\beta_1 = (i, 1)t_i \beta_2 = (i, 2)\beta_2 \text{ but } (i, 1)\beta_1 = (i, 1)t_j \beta_2 = (i, 1)\beta_2.$$

Therefore the two edges are labelled by a generator of  $\mathcal{T}_n$ . This means

$$(3.4) \quad \beta_1 = t_i \beta_2$$

for some  $t_i$ . By Remark 3.4, there is only one edge connecting  $\beta_1$  and  $\beta_2$  satisfying (3.4). So no two distinct edges share two vertices in  $\mathcal{C}_n$ .  $\square$

**Lemma 3.11.** *The graph  $X_1$  is simply connected.*

*Proof.* Suppose  $\ell$  is a loop in  $X_1$ . Consider the image of  $\ell$  under a Morse function  $h$ . If  $h(\ell) = [a, b]$  and  $b - a < 1$  then  $\ell$  lies entirely on a ball of a vertex with radius  $< 1$ . This ball is contractible and so  $\ell$  is null homotopic. Assume  $b - a \geq 1$ . We may further assume that  $a$  and  $b$  are integers. Let  $\ell \cap h^{-1}(a) = \{\alpha_1, \dots, \alpha_k\}$ . By Remark 3.4, there exists a unique edge  $e_j$  emanating from each  $\alpha_j$  ( $1 \leq j \leq k$ ). Observe that the loop  $\ell$  contains those edges and that  $\ell$  is homotopy equivalent to  $\ell - \cup_j e_j$  (Indeed, what we remove is not a whole edge but an edge minus the top vertex). Under this homotopy equivalence one can homotope  $\ell$  to another loop  $\ell'$  such that  $h(\ell') = [a + 1, b]$ . Essentially, in the passage from  $\ell$  to  $\ell'$ , we removed back-trackings involving vertices  $\alpha_1, \dots, \alpha_k$ . Apply the same process to establish homotopy equivalence between  $\ell$  and a vertex with height  $b$ . It can be shown that  $h^{-1}(b)$  is a singleton set even if it is not clear a priori. After applying the above homotopy equivalences up to  $b - a$  steps one obtains homotopy equivalence between the loop  $\ell$  and  $\ell \cap h^{-1}(b)$ . If the preimage  $h^{-1}(b)$  consisted of multiple number of vertices then  $\ell$  was homotopic to a discrete set  $h^{-1}(b)$ . We have shown that  $\ell$  is null homotopic.  $\square$

**Lifting across a square.** Fix  $n \geq 2$ . Suppose  $\beta$  is a vertex of  $X_n$ . For a pair of generators  $t_i$  and  $t_j$  ( $i \neq j$ ), there exists a unique square  $\sigma$  containing four vertices  $\beta, t_i\beta, t_j\beta$  and  $t_it_j\beta$ . Consider the path  $p$  with length 2, joining two vertices  $t_i\beta$  and  $t_j\beta$ , consisting of two lower edges of the square  $\sigma$ . We say a path  $p$  has a *turn* at  $\beta$ . This path  $p$  is homotopic to a path  $\tilde{p}$  rel two ends vertices which consists of two upper edges of  $\sigma$ . Figure 3.5 illustrates this idea. We say  $\tilde{p}$  is a *lifting* of  $p$  (across a square  $\sigma$ ). The notion of lifting is useful in proving Lemma 3.12 as well as amenability of  $\mathcal{C}_n$  in Section 3.5.

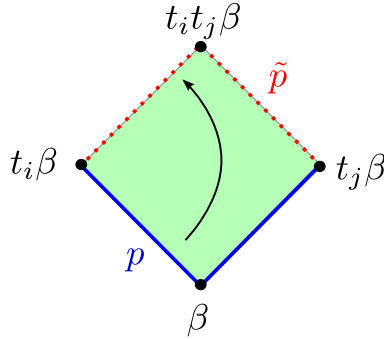


FIGURE 3.5. A path with a turn at  $\beta$  lifts to a path  $\tilde{p}$ .

**Lemma 3.12.** *For all  $n \geq 2$ ,  $X_n$  is simply connected.*

*Proof.* Suppose  $\ell : \mathbb{S}^1 \rightarrow X_n$  is a loop in  $X_n$  for  $n \geq 2$ . We may assume that  $\ell$  lies in 1-skeleton of  $X_n$  and that  $h(\ell) = [a, b]$  for some non negative integers  $a < b$ . We may further assume that  $\ell$  does not contain back-trackings. As in the proof of Lemma 3.11, one can eliminate the case  $b - a < 1$  by using rather trivial homotopy equivalence for a ball of a vertex with radius  $< 1$ . Let  $\alpha_1, \dots, \alpha_k$  be the vertices of  $\ell$ . By Proposition 3.7, there exists a unique  $\text{lub } \beta = \text{lub}\{\alpha_1, \dots, \alpha_k\}$ . We want to establish a homotopy equivalence between  $\ell$  and the trivial loop  $\beta$  by applying liftings across squares. We start with vertices in  $\ell \cap h^{-1}(a) = \{\beta_1, \dots, \beta_{k'}\}$ . The loop  $\ell$  may intersect multiple number of squares even for a single vertex  $\beta_j \in h^{-1}(a)$ . Let  $\ell^{-1}(\beta_j) = \{v_{j,1}, \dots, v_{j,r_j}\} \subset \mathbb{S}^1$  for  $j = 1, \dots, k'$ . From the

fact that  $\ell$  does not contain back-tracking, we have the following crucial observation for each  $j$ .

- (1) Each preimage  $v_{j,m}$  belongs to an interval  $I_{j,m}$  such that  $\ell(I_{j,m})$  is a path having a turn at  $\beta_j$ ,
- (2) the two end vertices of  $\ell(I_{j,m})$  are given by  $t_i\beta_j$  and  $t_{i'}\beta_j$  for distinct generators  $t_i$  and  $t_{i'}$  of  $\mathcal{T}_n$ .

So each interval  $I_{j,m}$  determines a square  $\sigma_{j,m}$  whose bottom vertex is  $\beta_j$ . Now take liftings of  $\ell(I_{j,m})$  for all  $j$  and  $m$  ( $1 \leq j \leq k', 1 \leq m \leq r$ ). The resulting loop  $\ell'$  is homotopy equivalent to  $\ell$ . Another important observation is that the top vertex  $w_{j,m}$  of a square  $\sigma_{j,m}$  in which lifting occurred satisfies

$$(3.5) \quad \beta \geq w_{j,m}.$$

Indeed, three vertices on  $\ell(I_{j,m})$  were already vertices of the loop  $\ell$ . The top vertex  $w_{j,m}$  is the *lub* of those three vertices. By Remark 3.8, each vertex  $w_{j,m}$  satisfies  $\beta \geq w_{j,m}$ . This means that in the passage from  $\ell$  to  $\ell'$  vertices of  $\ell \cap h^{-1}(a)$  were replaced by some vertices satisfying equation (3.5). So  $\beta$  is still an upper bound of the vertices of  $\ell'$ . Observe that  $h(\ell') = [a+1, b]$ .

Apply the same process consecutively to vertices with smallest height in each step. The property given by equation (3.5) ensures that this procedure stops after finitely many ( $h(\beta) - a$ ) steps. In all, the given loop  $\ell$  converges to  $\beta$ .  $\square$

**3.2. Properties of the monoid  $\mathcal{M}_n$ .** In this section we study some algebraic properties which are useful for us to study geometry of  $X_n$  in the sequel.

**Lemma 3.13.**  $\mathcal{M}_n = \mathcal{T}_n \mathcal{H}_n = \{tg \mid t \in \mathcal{T}_n, g \in \mathcal{H}_n\}$

*Proof.* Suppose  $\beta \in \mathcal{M}_n$  and  $\varphi(\beta) = (m_1, \dots, m_n)$ . There exist  $t \in \mathcal{T}_n$  and  $g \in \mathcal{H}_n$  such that  $h(t) = h(\alpha)$  and  $\varphi(\beta) = \varphi(tg)$ . Thus  $\beta$  and  $tg$  agree on  $Y_n - K$  for some finite set  $K$ . We want to find  $f \in \mathcal{H}_n$  such that  $\beta = tgf$ . Existence of such  $f$  comes from the fact that being injections,  $\beta$  and  $tg$  have right inverses  $f_1$  and  $f_2$  respectively (i.e., left inverses in the composition of functions). Consider those right inverses  $f_1$  and  $f_2$

$$f_1 : (K)\beta \rightarrow K \text{ and } f_2 : (K)tg \rightarrow K.$$

By definition,  $\beta f_1 = tg f_2$  on  $K$ . Observe that one can turn  $f_2 f_1^{-1}$  into an element  $f \in \mathcal{H}_n$  with  $\text{supp}(f) \subset (K)tg \cup (K)\beta$ . For example, one can extend  $f_2 f_1^{-1} : (K)tg \rightarrow (K)\beta$  to  $f : (K)tg \cup (K)\beta \rightarrow (K)tg \cup (K)\beta$  by a bijection  $f' : (K)\beta - (K)tg \rightarrow (K)tg - (K)\beta$ . More precisely,  $f$  is defined by

$$f = \begin{cases} f_2 f_1^{-1} & \text{on } (K)tg \\ f' & \text{on } (K)\beta - (K)tg \end{cases}$$

where  $f'$  is a bijection between congruent finite sets  $(K)\beta - (K)tg$  and  $(K)tg - (K)\beta$ .  $\square$

**Definition 3.14 (Greatest lower bound).** An element  $\beta$  is called a *lower bound* of  $\alpha_1$  and  $\alpha_2$  if it satisfies the first condition below, and is called the *greatest lower bound* (simply *glb*) if it satisfies the second condition as well.

- $\beta \leq \alpha_1$  and  $\beta \leq \alpha_2$ ,
- If  $\beta' \leq \alpha_1$  and  $\beta' \leq \alpha_2$  then  $\beta' \leq \beta$ .

The *glb* of  $\alpha_1$  and  $\alpha_2$  is denoted by  $\alpha_1 \vee \alpha_2$ .

Note that not every pair of vertices admits a lower bound (and in particular greatest lower bound). For example, the right figure in Figure 3.6 describes  $\alpha_1$  and  $\alpha_3$  which do not have a lower bound. If there was a common lower bound  $\gamma'$  then we would be forced to have  $(2, 1)\gamma' = (1, 1)\gamma'$ , contradicting the injectivity of  $\gamma'$ . The geometric interpretation of this is that there is no square containing  $\alpha_1$  and  $\alpha_3$ . However, if one replaces  $\alpha_1$  by  $\alpha_2$  then there exists a common lower bound (indeed  $\alpha_2 \vee \alpha_3$ ) for  $\alpha_2$  and  $\alpha_3$  as illustrated in the left figure. Note that  $(1, 1)\alpha_2 \neq (2, 1)\alpha_3$  but  $(1, 1)\alpha_1 = (1, 1) = (2, 1)\alpha_3$ . Lemma 3.22 states that such equation determines the existence of *glb* of those pairs. In general, existence of a common lower bound guarantees the existence of *glb*.

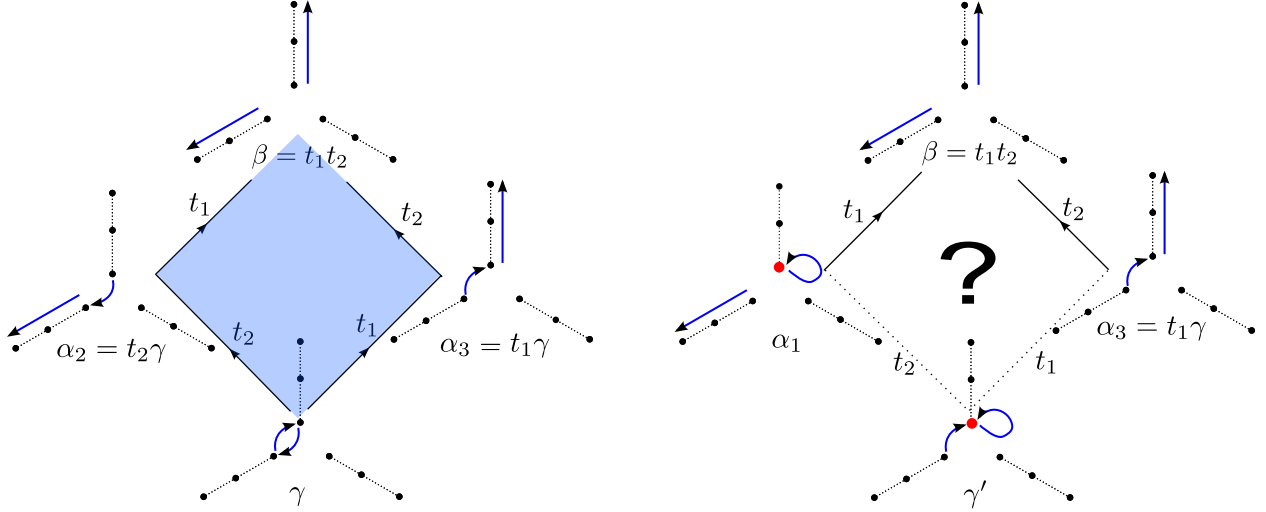


FIGURE 3.6. Examples  $\alpha_1, \alpha_2, \alpha_3$  where  $\alpha_2 \vee \alpha_3 = \gamma$  exists, but  $\alpha_1 \vee \alpha_3$  does not exist.

**Lemma 3.15.** *For any  $\alpha_1, \alpha_2 \in \mathcal{M}_n$ ,  $\alpha_1 \vee \alpha_2$  exists and unique if there is a lower bound of  $\alpha_1$  and  $\alpha_2$ .*

*Proof.* Suppose  $\gamma$  is a lower bound of  $\alpha_1$  and  $\alpha_2$ . Then there exist  $\tau = t_1^{k_1}, \dots, t_n^{k_n}$  and  $\tau' = t_1^{k'_1}, \dots, t_n^{k'_n}$  satisfying

$$\alpha_1 = \tau\gamma \text{ and } \alpha_2 = \tau'\gamma.$$

**Claim:**  $\beta := t_1^{m_1} \dots t_n^{m_n} \gamma$  is the *glb*, where  $m_i = \min\{k_i, k'_i\}$ .

The commutativity of  $\mathcal{T}_n$  implies that

$$\alpha_1 = t_1^{l_1} \dots t_n^{l_n} \beta \text{ and } \alpha_2 = t_1^{l'_1} \dots t_n^{l'_n} \beta$$

where  $l_i = \max\{k_i - k'_i, 0\}$  and  $l'_i = \max\{k'_i - k_i, 0\}$ . So  $\beta$  satisfies the first condition. Observe that  $l_i l'_i = 0$  for each  $i$ . This means that either

$$\alpha_1|_{R_i} = \beta|_{R_i} \text{ or } \alpha_2|_{R_i} = \beta|_{R_i}$$

for each  $i$ . Suppose  $\beta'$  is another element with  $\alpha_1 = \tau_1 \beta'$  and  $\alpha_2 = \tau_2 \beta'$  for some  $\tau_1, \tau_2 \in \mathcal{T}_n$ . Then we have

$$(\tau_1 \beta')|_{R_i} = \beta|_{R_i} \text{ or } (\tau_2 \beta')|_{R_i} = \beta|_{R_i}$$

for each  $i$ . So  $\beta' \leq \beta$ . Uniqueness of the *glb* follows from the second condition in Definition 3.14 together with antisymmetry of  $\geq$  (Proposition 3.1).  $\square$

**Corollary 3.16.** *If  $\alpha_1, \alpha_2 \in \mathcal{T}_n$  then  $\alpha_1 \vee \alpha_2$  always exists.*

*Proof.* If two elements  $\alpha_1$  and  $\alpha_2$  belong to  $\mathcal{T}_n$ , then they have a common lower bound  $1_{\mathcal{M}_n}$ , namely the identity map on  $Y_n$ . By Lemma 3.15,  $\alpha_1 \vee \alpha_2$  exists.  $\square$

**Lemma 3.17.** *For a pair of elements  $\alpha_1, \alpha_2 \in \mathcal{M}_n$ , the lub and glb of the pair can be characterized by height function as follows.*

- (1)  $\beta$  is the lub of  $\alpha_1, \alpha_2$  if  $\beta$  is an upper bound and  $\beta$  has the smallest height among the upper bounds;
- (2)  $\beta$  is the glb of  $\alpha_1, \alpha_2$  if  $\beta$  is a lower bound and  $\beta$  has the largest height among the lower bounds.

*Proof.* Suppose  $\beta$  is an element satisfying the condition (1) above. We want to show  $\beta = \alpha_1 \wedge \alpha_2$ . By the definition of  $\alpha_1 \wedge \alpha_2$  it is obvious that  $h(\beta) = h(\alpha_1 \wedge \alpha_2)$ . By Lemma 3.15,  $\beta \vee (\alpha_1 \wedge \alpha_2)$  exists since  $\beta$  and  $\alpha_1 \wedge \alpha_2$  have common lower bounds  $\alpha_1$  and  $\alpha_2$ . Note that  $\beta \vee (\alpha_1 \wedge \alpha_2)$  is an upper bound of  $\alpha_1$  and  $\alpha_2$ . If  $\beta \neq \alpha_1 \wedge \alpha_2$  then  $h(\beta \vee (\alpha_1 \wedge \alpha_2)) < h(\beta)$ , contradicting to our choice of  $\beta$ . So  $\beta = \alpha_1 \wedge \alpha_2$ . Analogous argument can be applied to the case of glb. One can draw a contradiction by assuming  $\beta \neq (\alpha_1 \vee \alpha_2)$  for an element  $\beta$  satisfying the condition of (2) above.  $\square$

The following specifies the relationship between *lub* and *glb* under assumption on existence of *glb*.

**Proposition 3.18.** *Suppose, for  $\alpha_1, \alpha_2 \in \mathcal{M}_n$ ,  $\tau\alpha_1 = \alpha_1 \wedge \alpha_2 = \tau'\alpha_2$  where  $\tau = t_1^{k_1} \cdots t_n^{k_n}$  and  $\tau' = t_1^{k'_1} \cdots t_n^{k'_n}$ . Then  $\alpha_1 \vee \alpha_2$  exists if and only if the following conditions are satisfied for all  $i = 1, 2, \dots, n$*

- *there is no common generator  $t_i$  in  $\tau$  and  $\tau'$ , i.e.,  $k_i k'_i = 0$ ,*
- $(\bigcup_{p \leq k_i} (i, p)\alpha_1) \cap (\bigcup_{p \leq k'_i} (i, p)\alpha_2) = \emptyset$ .

Moreover, we have

$$(3.6) \quad \tau'(\alpha_1 \vee \alpha_2) = \alpha_1 \text{ and } \tau(\alpha_1 \vee \alpha_2) = \alpha_2.$$

*Proof.* First we construct  $\gamma = \alpha_1 \vee \alpha_2$  by using the conditions. Define  $\gamma$  by

$$(i, p)\gamma = \begin{cases} (i, p)\alpha_1 & \text{if } k'_i = 0 \\ (i, p)\alpha_2 & \text{if } k_i = 0 \end{cases}$$

for each  $i = 1, \dots, n$ . This is well defined since  $k_i k'_i = 0$ , and if  $k_1 = k'_1 = 0$  then we have  $(i, p)\alpha_1 = (i, p)\tau\alpha_1 = (i, p)\tau'\alpha_2 = (i, p)\alpha_2$  for all  $p \in N$ . One can verify  $\gamma$  is the desired element as follows.

**Claim 1:**  $\tau'\gamma = \alpha_1$  and  $\tau\gamma = \alpha_1$ .

Fix  $i$ . Suppose  $k'_i = 0$ . For all  $p \in N$ , we have  $(i, p)\tau'\gamma = (i, p)t_i^{k'_i}\gamma = (i, p)\gamma = (i, p)\alpha_1$ . Suppose  $k'_i \neq 0$ . This means  $k_i = 0$  and so we have

$$\begin{aligned} (i, p)\tau'\gamma &= (i, p)t_i^{k'_i}\gamma = (i, p + k'_i)\gamma = (i, p + k'_i)\alpha_2 = (i, p)t_i^{k'_i}\alpha_2 \\ &= (i, p)\tau'\alpha_2 = (i, p)\tau\alpha_1 = (i, p)t_i^{k_i}\alpha_1 = (i, p)\alpha_1, \end{aligned}$$

for all  $p \in N$ . Since the above identities hold for every  $i$ , we have verified  $\tau'\gamma = \alpha_1$ . A similar argument shows that  $\tau\gamma = \alpha_1$ .

**Claim 2:**  $\gamma \in \mathcal{M}_n$ . We need to verify injectivity. The image  $(Y_n)\gamma$  coincides with  $(Y_n)\alpha_1$  up to a finite set. We already know that  $(i, p)\gamma = (i, p)\alpha_1$  if  $k'_i = 0$  ( $p \in \mathbb{N}$ ). If  $k'_i \neq 0$  then  $(i, p+k'_i)\gamma = (i, p)\alpha_1$  ( $p \in \mathbb{N}$ ). It suffices to show, for  $i$  with  $k'_i \neq 0$ , that  $(i, p)\gamma \notin (Y_n)\alpha_1$  for all  $p \leq k'_i$ . This is clear from the second condition of the Lemma because  $(i, p)\gamma \in \bigcup_{p \leq k'_i} (i, p)\alpha_2$ .

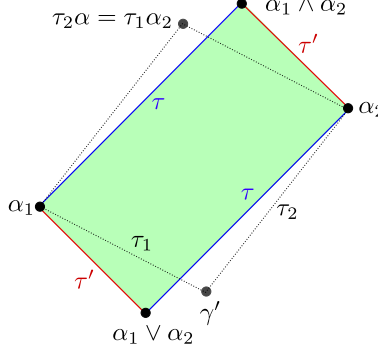


FIGURE 3.7. ‘Big rectangle’: if  $\alpha_1 \vee \alpha_2$  exists  $\alpha_1 \wedge \alpha_2$  determines  $\alpha_1 \vee \alpha_2$ , vice versa.

**Claim 3:**  $\gamma = \alpha_1 \vee \alpha_2$ . The lower bound  $\gamma$  has the largest height among lower bounds of  $\alpha_1$  and  $\alpha_2$ . If there was another lower bound  $\gamma'$  of  $\alpha_1$  and  $\alpha_2$  with  $h(\gamma') > h(\gamma)$  then there would be  $\tau_1, \tau_2 \in \mathcal{T}_n$  such that

$$\tau_1 \gamma' = \alpha_1, \tau_2 \gamma' = \alpha_2 \text{ and } |\tau_1| < |\tau'|.$$

This implies  $\alpha_1$  and  $\alpha_2$  have an upper bound  $\tau_2 \alpha_1 = \tau_1 \alpha_2$  such that  $h(\tau_1 \alpha_2) = (\tau_1) + h(\alpha_2)$  is strictly smaller than  $h(\alpha_1 \wedge \alpha_2) = h(\tau') + h(\alpha_2)$  (see Figure 3.7). This contradicts the fact that  $\alpha_1 \wedge \alpha_2$  has the smallest height among upper bounds of  $\alpha_1$  and  $\alpha_2$  (Lemma 3.17). So the lower bound  $\gamma$  attains maximum height among lower bounds of  $\alpha_1$  and  $\alpha_2$ . By Lemma 3.17,  $\gamma = \alpha_1 \vee \alpha_2$ .

The converse statement is rather easy to check. Suppose that  $\alpha_1 \vee \alpha_2$  exists and that

$$(3.7) \quad \tau_1(\alpha_1 \vee \alpha_2) = \alpha_1 \text{ and } \tau_2(\alpha_1 \vee \alpha_2) = \alpha_2.$$

for some  $\tau_1, \tau_2 \in \mathcal{T}_n$ . As in the proof of Lemma 3.15, we see that there is no common generator in  $\tau_1$  and  $\tau_2$ . So the first condition is satisfied. Injectivity of  $\gamma$  together with equation (3.7) guarantees the second condition.

Finally, the argument in Claim 3 above says that  $\alpha_1 \vee \alpha_2$  determines  $\alpha_1 \wedge \alpha_2$ . If  $\alpha_1 \vee \alpha_2$  satisfies equation (3.7) then we have

$$\tau_1 \tau = \tau_2 \tau'.$$

Now the condition that there is no common generator in pairs  $\tau$  and  $\tau'$ , and  $\tau_1$  and  $\tau_2$  enables one to conclude that  $\tau_1 = \tau'$  and  $\tau_2 = \tau$ . Hence equation (3.6) follows.  $\square$

**Definition 3.19 (Maximal elements).** For  $\beta \in \mathcal{M}_n$ ,  $\alpha$  is called a *maximal element* of  $\beta$  if it satisfies

$$(3.8) \quad \beta = t_i \alpha$$

for some generator  $t_i$  of  $\mathcal{T}_n$ .

The following Lemma says the number of maximal elements of  $\beta$  varies depending on the height of  $\beta$ .



**Lemma 3.20.** *Suppose  $h(\beta) = h$ . There exists  $n \times h$  many maximal elements of  $\beta$ .*

*Proof.* Fix  $i$ . The identity (3.8) determines  $\alpha$  completely except  $(i, 1)\alpha$ . It also implies  $(Y_n - (i, 1))\alpha = (Y_n)\beta$ . So there are precisely  $|S(\beta)| = |Y_n - (Y_n)\beta| = h$  many choices for  $(i, 1)\alpha$ . It turns out any of these choice defines an injective map  $\alpha$ . Consider elements  $\alpha_1, \dots, \alpha_h$  defined by

$$(j, p)\alpha_k = \begin{cases} (j, p)\beta & \text{if } j \neq i, p \in \mathbb{N} \\ (i, p+1)\beta & \text{if } j = i, p \in \mathbb{N}, \end{cases}$$

and  $(i, 1)\alpha_k \in S(\beta)$ . They are all desired  $h$  many injective maps. □

**Remark 3.21.** The argument of Lemma 3.20 implies that, each maximal element  $\alpha_k$  of  $\beta$  is labelled by  $t_i$  as well as a point of  $S(\beta)$ . Figures 3.2 and 3.6 illustrate this idea for  $\beta = t_1 t_2$ . Note that  $S(\beta) = \{(1, 1), (1, 2)\}$ . In those figures maximal elements  $\alpha_1, \alpha_2$  and  $\alpha_3$  are labelled by

$$\alpha_1 \leftrightarrow (t_1, (1, 1)), \alpha_2 \leftrightarrow (t_1, (1, 2)), \alpha_3 \leftrightarrow (t_2, (1, 1)).$$

In general, for  $\beta \in \mathcal{M}_n$ , the set of maximal elements of  $\beta$  is in 1-to-1 correspondence with  $\{1, \dots, n\} \times S(\beta)$ .

The corollaries below follows from Proposition 3.18. They provide criterions when a collection of maximal elements form squares and  $k$ -cubes respectively.

**Corollary 3.22.** *Suppose  $t_i \alpha_1 = t_j \alpha_2$  ( $i \neq j$ ). There exists  $\alpha_1 \vee \alpha_2$  if and only if the two first coordinates are distinct and the second coordinates are distinct (using the ordered pair notation introduced in Remark 3.21).*

*Proof.* This is a special case of Proposition 3.18, with the two conditions of that proposition rephrased in terms of the coordinates introduced in Remark 3.21. □

**Corollary 3.23.** *Suppose  $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$  is a collection of maximal elements of  $\beta$ . There exists  $\text{glb}(\alpha_1, \dots, \alpha_k)$  if and only if the first coordinates are all distinct and the second coordinates are all distinct (using the ordered pair notation introduced in Remark 3.21).*

*Proof.* We construct  $\text{glb}$  inductively with the base case  $k = 2$ , which is done in Corollary 3.22. Assume the statement holds for  $k$  maximal vertices of  $\beta$ . Say the  $\text{glb}$  of those vertices is  $\gamma$ . Note that the information of  $k$  vertices encoded in the coordinates completely determine  $\gamma$ . Now apply Proposition 3.18 to  $\gamma$  and  $\alpha_{k+1}$  to check the existence of  $\gamma \vee \alpha_{k+1}$ . Note that  $\gamma \wedge \alpha_{k+1} = \beta$ . The condition on distinct first coordinates implies there is no common generator  $t_i$  in the two paths. The condition on the other coordinates implies the second condition of Proposition 3.18. So  $\gamma \vee \alpha_{k+1}$  exists.

Conversely, the existence of  $\text{glb}$  directly implies there exists a cube whose top vertex is  $\beta$ . So the condition on coordinates is satisfied. □

**3.3.  $X_n$  is CAT(0).** In this section we show  $X_n$  is a cubing for each  $n \in \mathbb{N}$ , i.e.,  $X_n$  is a 1-connected non positively curved cubical complex. There are several ways to think about this result.

By Lemma 3.12,  $X_n$  is simply connected. In Subsection 3.4, we prove  $X_n$  is non positively curved by using Gromov's *link condition*.

Another way to see  $X_n$  is a cubing is to use the fact ([7], [15]) that there is 1-to-1 correspondence

$$\text{the class of cubings} \leftrightarrow \text{the class of median graphs}$$

One associates a median graph to a cubing by considering the 1-skeleton of the cubing. In the reverse direction, one thinks of a median graph as the 1-skeleton of a cubing, and defines the cubing inductively on skeleta, as in Definition 3.9. In Subsection 3.5 we prove  $\mathcal{C}_n$  is a median graph for all  $n \in \mathbb{N}$ .

In [17], Sageev provided explicit construction of a cubing associated to a multi-ended group. In Subsection ??, we discuss an interpretation of  $X_n$  in terms of Sageev's construction.

### 3.4. Gromov link condition.

**Definition 3.24 (cubical complex).** Intuitively, a cubical complex is a regular CW-complex, except that it is built out of Euclidean cubes  $I^k$  instead of balls. More precisely, a *cubical complex*  $X$  is a CW-complex where for each  $k$ -cell  $\sigma_j^k \subset X$  its attaching map  $\varphi_j : \partial I^k \rightarrow X^{n-1}$  satisfies the following conditions:

- (1) the restriction of  $\varphi_j$  to each face of  $I^k$  is a linear homeomorphism onto a cube of one lower dimension,
- (2)  $\varphi_j$  is a homeomorphism onto its image.

We give  $X$  the standard CW-topology.

The non-positive curvature condition we will use is a local condition captured by conditions on the link of a vertex.

**Definition 3.25 (Link of a vertex).** The *link of a vertex  $v$  in  $I^k$* , denoted by  $Lk(v, I^k)$ , is defined to be intersection of the cube  $I^k$  and the unit sphere  $\mathbb{S}^{k-1}$  centered at  $v$  with respect to  $L^1$  metric. Note that  $Lk(v, I^k)$  is the standard simplex of dimension  $k - 1$ . For a vertex  $\alpha \in X$  and a cell  $\sigma_j^k$  containing  $\alpha$ , the *link of  $\alpha$  in  $\sigma_j^k$*  is defined to be the image  $\varphi_j(Lk(v, I^k))$ , where  $\varphi_j(v) = \alpha$ . The *link of  $\alpha$  in  $X$* , denoted by  $Lk(\alpha, X)$ , is defined to be the union of all links of  $\alpha$  in cells containing  $\alpha$ .

**Definition 3.26 (Ascending/Descending links of a vertex).** For a vertex  $\alpha$  of a cubical complex  $X$  equipped with a Morse function  $h$ , the descending link  $Lk_\downarrow(\alpha, X)$  is defined by

$$Lk_\downarrow(\alpha, X) = \bigcup \{ \varphi_j(Lk(v, \sigma_j^k)) : h\varphi_j \text{ attains maximum at } v, \varphi_j(v) = \alpha \}.$$

Likewise, the ascending link  $Lk_\uparrow(\alpha, X)$  is defined by

$$Lk_\uparrow(\alpha, X) = \bigcup \{ \varphi_j(Lk(v, \sigma_j^k)) : h\varphi_j \text{ attains minimum at } v, \varphi_j(v) = \alpha \}.$$

**Definition 3.27 (Flag complex).** A simplicial complex  $L$  is said to be a *flag complex* if every collection of pairwise adjacent vertices of  $L$  spans a simplex of  $L$ .

**Definition 3.28 (Gromov's condition).** A cubical complex  $X$  satisfies the Gromov's condition if  $Lk(\alpha, X)$  is a flag complex for each  $\alpha \in X^{(0)}$ .

Following Gromov, we say that a cubical complex  $X$  is *non positively curved* (simply NPC) if it satisfies the Gromov's condition.

From the way we constructed  $X_n$ , we see that any vertex of  $Lk_\downarrow(\alpha, X_n)$  corresponds to a vertex  $\beta$  such that  $\alpha = t_i\beta$  for some generator  $t_i \in \mathcal{T}_n$ . By Remark 3.21, there exists a

bijection

$$\{\beta \mid \alpha = t_i \beta \text{ for some } t_i\} \leftrightarrow \{i \mid 1 \leq i \leq n\} \times S(\alpha)$$

Identify the later set with  $\{(i, k) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq n, 1 \leq k \leq h(\alpha)\}$ . Under this (composition of) identification, points  $\{(i, k) \mid 1 \leq k \leq h(\alpha)\}$  correspond to vertices in  $Lk_\downarrow(\alpha, X_n)$  which are joined to  $\alpha$  by edges labelled by  $t_i$ . Consider the simplicial complex  $L_{n,h}$  for  $h \in \mathbb{N}$  defined by

- (1) vertices:  $L_{n,h}^{(0)} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq n, 1 \leq k \leq h\}$
- (2) simplexes: A collection of vertices  $\{(x_0, y_0), \dots, (x_k, y_k)\}$  form a  $k$ -simplex if  $x_i$ 's are all distinct and  $y_i$ 's are all distinct.

**Remark 3.29.** By Corollary 3.23, we see that  $L_{n,h}$  and  $Lk_\downarrow(\alpha, X_n)$  are identical if  $h = h(\alpha)$ . Note that  $L_{n,h}$  is flag for all  $n, h \in \mathbb{N}$ . If two vertices of  $L_{n,h}$  are connected by an edge, their first coordinates are distinct and second coordinates are distinct. This means that any collection  $C \subset L_{n,h}^{(0)}$  of pairwise adjacent vertices satisfy the condition (2) of definition of  $L_{n,h}$  above. So a collection  $C$  forms a simplex of  $L_{n,h}$ .

**Lemma 3.30.** *For every vertex  $\alpha \in X_n$ ,  $Lk(\alpha, X_n)$  is flag.*

*Proof.* Suppose  $C$  is a collection of pairwise adjacent vertices of  $Lk(\alpha, X_n)$ . We may assume  $C = \{z_1, \dots, z_k\} \cup \{z'_1, \dots, z'_{k'}\}$  where  $z_i \in Lk_\uparrow(\alpha, X_n)$  and  $z'_j \in Lk_\downarrow(\alpha, X_n)$ . For each vertex  $\alpha \in X_n$ , there exists unique  $n$ -cube having  $\alpha$  as the bottom vertex. So  $Lk_\uparrow(\alpha, X_n)$  is simply the standard  $(n-1)$ -simplex. By Remark 3.29  $Lk_\downarrow(\alpha, X_n)$  is also flag. So those subcollections of  $C$  form a  $(k-1)$ -simplex  $\sigma$  and a  $(k'-1)$ -simplex  $\sigma'$  respectively in the ambient complexes.

Our claim is that there exist  $(k+k')$ -cube which contributes  $(k+k'-1)$  simplex  $\sigma * \sigma'$  to  $Lk(\alpha, X_n)$ . Consider the ordered pair notation introduced in Remark 3.21 for  $\{z_1, \dots, z_k\}$ . Let  $T_1 = \{t_{i_1}, \dots, t_{i_k}\} \subset \{t_1, \dots, t_n\}$  denote the set consisting of first coordinates of vertices  $\{z_1, \dots, z_k\}$ . Likewise let  $T_2 = \{t_{j_1}, \dots, t_{j_{k'}}\}$  be the set corresponding to  $\{z'_1, \dots, z'_{k'}\}$ . Observe that the  $(k-1)$ -simplex  $\sigma \subset Lk_\downarrow(\alpha, X_n)$  corresponds to  $k$ -cube  $\rho$  with top vertex  $\alpha$ , which is generated by  $T_1$ . Similarly,  $\sigma' \subset Lk_\uparrow(\alpha, X_n)$  corresponds to  $k'$ -cube  $\rho'$  which is generated by  $T_2$ , whose bottom vertex is  $\alpha$ . Since  $T_1$  and  $T_2$  disjoint, the  $k$ -cube  $\rho$  together with edges labelled by  $T_2$  spans a  $(k+k')$ -cube. This cube is the desired cube containing  $\rho$  and  $\rho'$ , and so  $\alpha$  as well. So a collection  $C$  forms a simplex  $\sigma * \sigma'$  in  $Lk(\alpha, X_n)$ .  $\square$

By Lemma 3.12 and Lemma 3.30 we have the following Theorem.

**Theorem 3.31.** *For all  $n \in \mathbb{N}$ ,  $X_n$  is a cubing.*

The following is known facts for a cubing, and proofs can be found in [3].

**Corollary 3.32.** *For all  $n \in \mathbb{N}$ ,  $X_n$  is uniquely geodesic and is contractible.*

**3.5. Cayley graph  $\mathcal{C}_n$  is a median graph.** For an edge  $e$  of  $\mathcal{C}_n$ , let  $\partial_- e$  and  $\partial_+ e$  denote the initial and terminal vertices of  $e$ . Although generators of  $\mathcal{T}_n$  do not have inverses, we can consider the *reverse* edge of  $e$ . Let  $\bar{e}$  denote the reverse edge of an edge  $e$ , i.e.,  $\partial_- \bar{e} = \partial_+ e$  and  $\partial_+ \bar{e} = \partial_- e$ . Note that  $\bar{\bar{e}} = e$ .

By a *path* we mean an edge path; concatenation of edges (including reverse edges)  $e_1, \dots, e_k$  where  $\partial_+(e_i) = \partial_-(e_{i+1})$  for  $i = 1, 2, \dots, k-1$ . If a path  $p$  is a concatenation of edges  $e_1, \dots, e_k$  in order, we write this path as  $p = e_1 \cdots e_k$  and the *reverse* path  $\bar{p}$  of  $p$  is defined by  $\bar{p} = \bar{e}_k \cdots \bar{e}_1$ .

A path is called *ascending* if it does not contain reverse edges. Similarly a path is called *descending* if it consists of only reverse edges. Obviously the reverse path of an ascending path is descending and vice versa. Suppose  $p$  is an ascending path joining  $\alpha$  to  $\tau\alpha$  for some  $\tau \in \mathcal{T}_n$ . One reads the edge labeling of  $p$  in order to form a word  $\tau = \tau_1 \cdots \tau_k$ ,  $\tau_i \in \{t_1, \dots, t_n\}$ . The expression for  $\tau$  is not unique in  $(\mathcal{T}_n)$  because of commutativity of  $\mathcal{T}_n$ . Indeed permutations of generators in  $\tau$  produce a class of paths rel two end vertices  $\alpha$  and  $\tau\alpha$ . In the sequel, a choice of such path is less relevant. Instead we will be interested in two end vertices of an ascending path. When we say a path  $p$  given by  $\tau \in \mathcal{T}_n$ , we mean a choice from the class of paths rel two end vertices  $\alpha$  and  $\tau\alpha$ , and write  $p \simeq \tau$ .

For a pair of vertices  $\alpha$  and  $\beta$ , the *distance* in  $\mathcal{C}_n$  is the smallest length of paths joining them. We denote the distance by  $d(\cdot, \cdot)$ . A *geodesic*  $[\alpha, \beta]$  joining vertices  $\alpha$  and  $\beta$  is a path whose length is  $d(\alpha, \beta)$ .

**Remark 3.33.** Note that  $d(\alpha, \beta) \geq |h(\alpha) - h(\beta)|$ . Every ascending/descending path is a geodesic. If  $p$  is an ascending/descending path joining  $\alpha$  to  $\beta$  then  $d(\alpha, \beta) = |h(\beta) - h(\alpha)|$ .

### Standard geodesics in $\mathcal{C}_n$ .

We say a path  $p$  has a *turn* if  $p$  contains  $\bar{e}_{t_i} e_{t_j}$  for some  $i, j$ . One can apply liftings (finitely many times) to transform  $p$  to a path of the form

$$(3.9) \quad e_{t_{i_1}} \cdots e_{t_{i_k}} \bar{e}_{t_{j_1}} \cdots \bar{e}_{t_{j_\ell}} \quad (k, \ell \geq 0).$$

Such path is called *standard*, i.e., a path is standard if it is a concatenation of one ascending path and one descending path where the ascending path occurs first. Figure 3.8 illustrates liftings of paths. Note that the length of paths does not increase under liftings. As before, for a standard path  $p$  given by  $(\tau)(\bar{\tau}')$  joining  $\alpha$  to  $\beta$ , there are many expressions (determined by permutations in each parenthesis). However every such choice shares important information: initial vertex  $\alpha$ , terminal vertex  $\beta$  and the top vertex. We write  $p \simeq \tau \cdot \bar{\tau}'$  and we mean  $p$  is a concatenation of one ascending path and one descending path which are determined by choices in  $\tau$  and  $\tau'$  respectively.

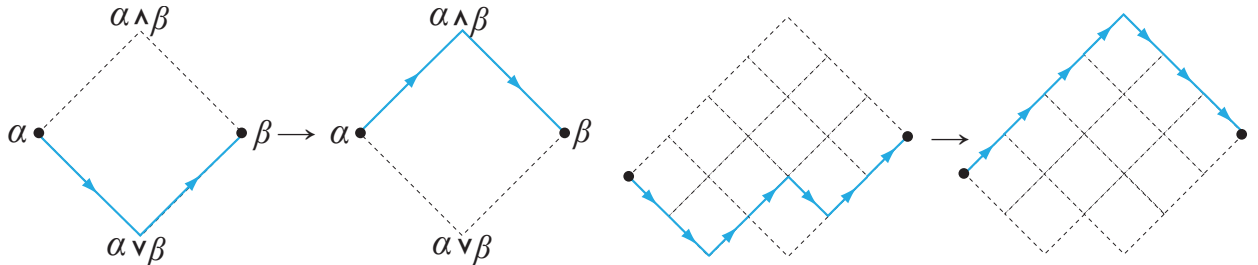


FIGURE 3.8. Every path can be transformed to a standard path via liftings

**Remark 3.34.** Note that the top vertex of a standard path is a common upper bound of the two end vertices. By Lemma 3.17, a standard path  $\pi \cdot \bar{\pi}'$  joining  $\alpha$  to  $\beta$  is a geodesic if and only if  $\pi\alpha = \alpha \wedge \beta = \pi'\beta$ .

A standard path is called a *standard geodesic* if it is a geodesic.

**Definition 3.35 (Median graph).** The *geodesic interval*  $\mathcal{I}[\alpha, \beta]$  is the collection of vertices lying on geodesics  $[\alpha, \beta]$ . A graph is called a *median* if, for each triple of vertices  $\alpha, \beta, \gamma$ , the geodesic intervals  $[\alpha, \beta]$ ,  $[\beta, \gamma]$  and  $[\gamma, \alpha]$  have a unique common point.

**Proposition 3.36.**  $\gamma \in \mathcal{I}[\alpha, \beta]$  if and only if  $d(\alpha, \beta) = d(\alpha, \gamma) + d(\gamma, \beta)$ .

*Proof.* Suppose  $\gamma \in \mathcal{I}[\alpha, \beta]$ . There exists a geodesic joining  $\alpha$  and  $\beta$  that contains  $\gamma$ . So  $\gamma$  determines two subpaths which are geodesics joining  $\alpha$  to  $\gamma$ , and  $\gamma$  to  $\beta$  respectively. Thus  $d(\alpha, \beta) = d(\alpha, \gamma) + d(\gamma, \beta)$ . Conversely, if  $d(\alpha, \beta) = d(\alpha, \gamma) + d(\gamma, \beta)$  then any concatenation of two geodesics  $[\alpha, \gamma]$  and  $[\gamma, \beta]$  is again a geodesic containing  $\gamma$  and hence  $\gamma \in \mathcal{I}[\alpha, \beta]$ .  $\square$

**Lemma 3.37.** If  $\gamma \in \mathcal{I}[\alpha, \beta]$  then

$$\mathcal{I}[\alpha, \beta] \cap \mathcal{I}[\beta, \gamma] \cap \mathcal{I}[\gamma, \alpha] = \{\gamma\}.$$

*Proof.* Obviously  $\gamma$  is in the intersection if  $\gamma \in \mathcal{I}[\alpha, \beta]$ . Suppose the intersection contains other point  $\gamma'$ . From Proposition 3.36, we have

$$\begin{aligned} d(\alpha, \beta) &= d(\alpha, \gamma) + d(\gamma, \beta) \\ &= d(\alpha, \gamma') + d(\gamma', \gamma) + d(\gamma, \gamma') + d(\gamma', \beta) \quad (\gamma' \in \mathcal{I}[\alpha, \gamma], \gamma' \in \mathcal{I}[\beta, \gamma]) \\ &= d(\alpha, \gamma') + d(\gamma', \beta) + 2d(\gamma', \gamma) \quad (\gamma' \in \mathcal{I}[\alpha, \beta]) \\ &= d(\alpha, \beta) + 2d(\gamma', \gamma). \end{aligned}$$

So  $d(\gamma', \gamma) = 0$  or  $\gamma' = \gamma$ .  $\square$

We will need the following two lemmas.

**Lemma 3.38.** Suppose  $\gamma, \gamma' \in \mathcal{I}[\alpha, \beta]$ . Then  $\gamma \wedge \gamma' \in \mathcal{I}[\alpha, \beta]$ . Moreover if there exists a lower bound of  $\gamma$  and  $\gamma'$  then  $\gamma \vee \gamma' \in \mathcal{I}[\alpha, \beta]$

*Proof.* Suppose  $\gamma, \gamma' \in \mathcal{I}[\alpha, \beta]$ . We first construct a geodesic joining  $\alpha$  to  $\beta$  which passes  $\gamma \wedge \gamma'$ . Consider elements  $\pi, \pi', \tau, \tau', \sigma, \sigma', \rho, \rho' \in \mathcal{T}_n$  to express  $\alpha \wedge \gamma, \alpha \wedge \gamma', \gamma \wedge \beta$ , and  $\gamma' \wedge \beta$  as follows

$$\pi\alpha = \alpha \wedge \gamma = \tau\gamma, \pi'\alpha = \alpha \wedge \gamma' = \tau'\gamma', \sigma\gamma = \gamma \wedge \beta = \rho\beta, \text{ and } \rho'\gamma' = \gamma' \wedge \beta = \rho'\beta.$$

By Remark 3.34, the following standard paths are all geodesic:  $\pi \cdot \bar{\tau}$  joining  $\alpha$  and  $\gamma$ ;  $\sigma \cdot \bar{\rho}$  joining  $\gamma$  and  $\beta$ ;  $\pi' \cdot \bar{\tau}'$  joining  $\alpha$  and  $\gamma'$ ;  $\sigma' \cdot \bar{\rho}'$  joining  $\gamma'$  and  $\beta$ . Figure 3.9 illustrates those standard paths.

**Claim 1:**  $\gamma \wedge \gamma' \leq \alpha \wedge \beta$ . The standard path  $\pi\bar{\tau}$  connecting  $\alpha$  to  $\gamma$  is a geodesic by Remark 3.34. By the same reason, the standard path  $\sigma\bar{\rho}$  connecting  $\gamma$  to  $\beta$  is a geodesic. Consider the path  $p_1$  defined by the concatenation of  $\pi \cdot \bar{\tau}$  and  $\sigma \cdot \bar{\rho}$ ,

$$p_1 \simeq \pi \cdot \bar{\tau} \cdot \sigma \cdot \bar{\rho}.$$

By Proposition 3.36, this path  $p_1$  is a geodesic since  $\gamma \in \mathcal{I}[\alpha, \beta]$ . Similarly we have a geodesic  $p_2$  of  $\alpha$  and  $\beta$  which passes  $\gamma'$  given by

$$p_2 \simeq \pi' \cdot \bar{\tau}' \cdot \sigma' \cdot \bar{\rho}'.$$

By applying liftings to  $p_1$  and  $p_2$  one obtains geodesics  $\tilde{p}_1$  and  $\tilde{p}_2$  given by

$$\tilde{p}_1 \simeq \pi \cdot \sigma \cdot \bar{\tau} \cdot \bar{\rho}, \quad \tilde{p}_2 \simeq \pi' \cdot \sigma' \cdot \bar{\tau}' \cdot \bar{\rho}'.$$

From Remark 3.34, we see that the top vertex of these paths is  $\alpha \wedge \beta$ . Thus  $\alpha \wedge \beta = \pi\sigma\alpha = \tau\sigma\gamma$  and  $\alpha \wedge \beta = \pi'\sigma'\alpha = \tau'\sigma'\gamma'$ . So  $\alpha \wedge \beta \geq \gamma$  and  $\alpha \wedge \beta \geq \gamma'$ , and Claim 1 is verified.

A chain  $\gamma \leq (\gamma \wedge \gamma') \leq (\alpha \wedge \beta)$  allows one to decompose  $\tau$  and  $\sigma$  as  $\tau = \tau_1\tau_2$  and  $\sigma = \sigma_1\sigma_2$  so that

$$(3.10) \quad \tau_1\sigma_1\gamma = \gamma \wedge \gamma', \quad \tau_2\sigma_2(\gamma \wedge \gamma') = \alpha \wedge \beta.$$

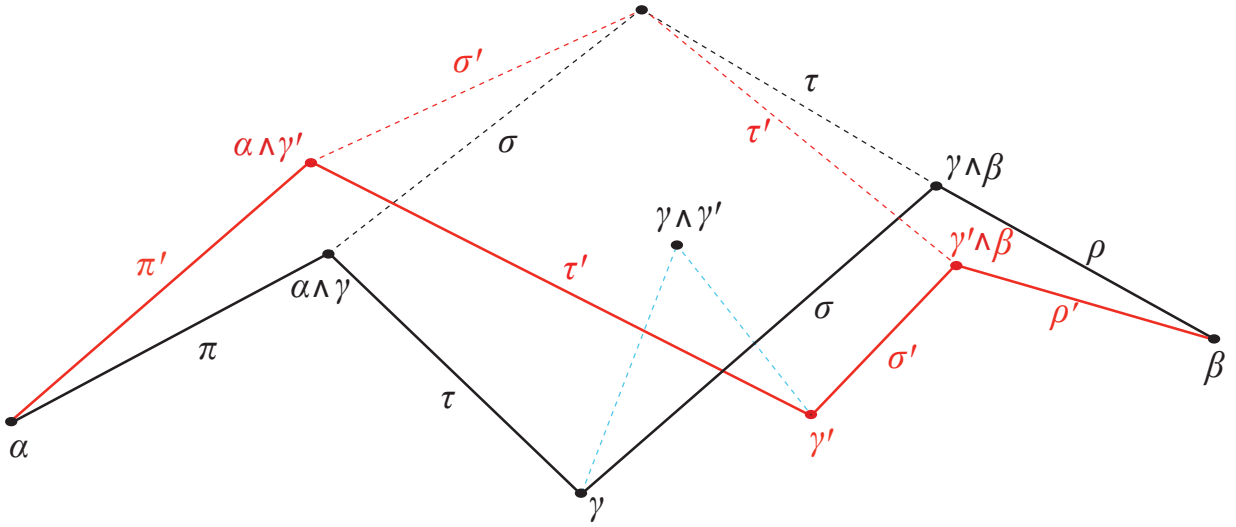


FIGURE 3.9. Two geodesics  $p_1 \simeq \pi \cdot \bar{\tau} \cdot \sigma \cdot \bar{\rho}$  and  $p_2 \simeq \pi' \cdot \bar{\tau}' \cdot \sigma' \cdot \bar{\rho}'$  of  $\alpha$  and  $\beta$  passing  $\gamma$  and  $\gamma'$  respectively

Figure 3.10 illustrates this decomposition. Consider the path connecting  $\alpha$  to  $\gamma \wedge \gamma'$  and then to  $\beta$  defined by

$$p_3 \simeq \pi \cdot \sigma_1 \cdot \bar{\tau}_2 \cdot \sigma_2 \cdot \bar{\tau}_1 \cdot \bar{\rho}.$$

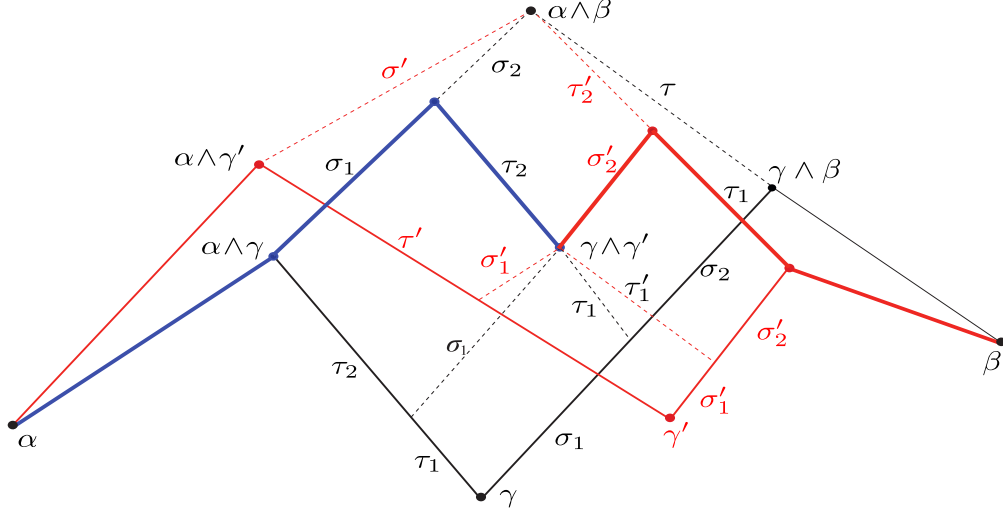


FIGURE 3.10. A geodesic  $p_3 \simeq \pi \cdot \sigma_1 \cdot \bar{\tau}_2 \cdot \sigma_2 \cdot \bar{\tau}_1 \cdot \bar{\rho}$  of  $\alpha$  and  $\beta$  passing  $\gamma \wedge \gamma'$ .

Apply appropriate liftings to the path  $p_3$  to obtain the geodesic  $\tilde{p}_1$ . Since liftings do not increase lengths, two paths  $\tilde{p}_1$  and  $p_3$  have the same length. Thus  $p_3$  is a geodesic joining  $\alpha$  to  $\beta$ , which passes  $\gamma \wedge \gamma'$ . Therefore  $\gamma \wedge \gamma' \in \mathcal{I}[\alpha, \beta]$ .

For the second assertion of the Lemma, we assume the existence of  $\gamma \vee \gamma'$ . As before, use a chain  $\gamma' \leq (\gamma \wedge \gamma') \leq (\alpha \wedge \beta)$  to get the decomposition of  $\tau'$  and  $\sigma'$ ;  $\tau' = \tau'_1 \tau'_2$ ,  $\sigma' = \sigma'_1 \sigma'_2$

such that

$$(3.11) \quad \tau'_1 \sigma'_1 \gamma = \gamma \wedge \gamma', \quad \tau'_2 \sigma'_2 (\gamma \wedge \gamma') = \alpha \wedge \beta$$

From identities (3.10) and (3.11) we have  $\tau_2 \sigma_2 = \tau'_2 \sigma'_2$ . Proposition 3.18 together with the existence of  $\gamma \vee \gamma'$  implies that

$$\tau'_1 \sigma'_1 (\gamma \vee \gamma') = \gamma, \text{ and } \tau_1 \sigma_1 (\gamma \vee \gamma') = \gamma'.$$

**Claim 2:**  $\tau_2 = \tau'_2$  and  $\sigma_2 = \sigma'_2$ . Look at the loop  $\ell_1$  formed by two ascending paths  $\tau\sigma$  and  $\sigma\tau$  emanating from  $\gamma$ . We first show  $\ell_1$  fits into the situation described in Proposition 3.18, i.e.,  $(\alpha \wedge \gamma) \wedge (\gamma \wedge \beta) = \alpha \wedge \beta$  and  $(\alpha \wedge \gamma) \vee (\gamma \wedge \beta) = \gamma$ . This loop has  $\gamma$  and  $\alpha \wedge \beta$  as bottom and top vertices respectively. Recall that the path  $p_1 \simeq \pi \cdot \bar{\tau} \cdot \sigma \cdot \bar{\rho}$  is a geodesic of  $\alpha$  and  $\beta$ . The restriction  $p_4$  of  $p_1$  defined by

$$p_4 \simeq \bar{\tau} \cdot \sigma$$

is also a geodesic joining  $\alpha \wedge \gamma$  and  $\gamma \wedge \beta$ . Consider the standard geodesic  $p_5$ , joining  $\alpha$  to  $\beta$ , defined by

$$p_5 \simeq \pi \cdot \sigma \cdot \bar{\tau} \cdot \bar{\rho}.$$

This geodesic  $p_5$  also passes  $\alpha \wedge \gamma$  and  $\gamma \wedge \beta$ . So the restriction  $p_6$  of  $p_5$  defined by

$$p_6 \simeq \sigma \cdot \bar{\tau}$$

is a standard geodesic connecting  $\alpha \wedge \gamma$  to  $\gamma \wedge \beta$ . By Remark 3.34,  $(\alpha \wedge \gamma) \wedge (\gamma \wedge \beta) = \alpha \wedge \beta$ . Moreover, by Proposition 3.18,  $(\alpha \wedge \gamma) \vee (\gamma \wedge \beta) = \gamma$ .

There is another loop  $\ell_2$  formed by two ascending paths  $\tau'\sigma'$  and  $\sigma'\tau'$  emanating from  $\gamma'$ . This loop  $\ell_2$  has  $\alpha \wedge \beta$  as top vertex and  $\gamma'$  as bottom vertex. By an analogous argument that we applied to the loop  $\ell_1$ , one can show

$$(\alpha \wedge \gamma') \wedge (\gamma' \wedge \beta) = \alpha \wedge \beta, \quad (\alpha \wedge \gamma') \vee (\gamma' \wedge \beta) = \gamma'.$$

Again, one can deduce the following fact from  $\ell_3$ , formed by two ascending paths  $\tau'_1 \sigma'_1 \tau\sigma$  and  $\tau_1 \sigma_1 \sigma' \tau'$  emanating from  $\gamma \vee \gamma'$ .

$$(\alpha \wedge \gamma) \wedge (\gamma' \wedge \beta) = \alpha \wedge \beta, \quad (\alpha \wedge \gamma) \vee (\gamma' \wedge \beta) = \gamma \vee \gamma'.$$

Now Proposition 3.18 implies that  $\sigma$  and  $\tau$  do not share a letter  $t_i$ . Similarly there is no common letter between  $\sigma'$  and  $\tau'$ , and  $\sigma$  and  $\tau'$ . So we have  $\tau_2 = \tau'_2$  and  $\sigma_2 = \sigma'_2$  from the identity  $\tau_2 \sigma_2 = \tau'_2 \sigma'_2$ .

**Claim 3**  $\tau_1 = \sigma'_1 = 1$ . Apply Proposition 3.18 to the loop  $\ell_3$  again to see that two diagonal edges of  $\ell_3$  are given by the same element of  $\mathcal{T}_n$ . In particular we have

$$(3.12) \quad \sigma = \tau_1 \sigma_1 \sigma'.$$

This means

$$\sigma = \sigma_1 \sigma_2 = \tau_1 \sigma_1 \sigma'_1 \sigma'_2$$

By Claim 2 we have  $\tau_1 \sigma'_1 = 1$ . This completes the proof since the path  $p \simeq \pi \bar{\tau} \bar{\tau}'_1 \sigma_1 \sigma' \bar{\rho}'$  passes  $\gamma \vee \gamma'$  and lifts to the geodesic  $p_3$ ;

$$\begin{aligned} p &\simeq \pi \bar{\tau}_2 \bar{\tau}'_1 \sigma_1 \sigma'_2 \bar{\rho}' \\ &\simeq \pi \bar{\tau}_2 \sigma_1 \bar{\tau}'_1 \sigma'_2 \bar{\rho}' \\ &\simeq \pi \sigma_1 \bar{\tau}_2 \sigma_2 \bar{\tau}'_1 \bar{\rho}'. \end{aligned}$$

□

**Corollary 3.39.** *For any vertices  $\alpha, \beta$ ,  $\mathcal{I}[\alpha, \beta]$  is convex in  $\mathcal{C}_n^{(0)}$ , i.e., for any  $\gamma, \gamma' \in \mathcal{I}[\alpha, \beta]$ ,  $\mathcal{I}[\gamma, \gamma'] \subset \mathcal{I}[\alpha, \beta]$ .*

*Proof.* In case of  $\gamma \leq \gamma'$ , one can show that  $\mathcal{I}[\gamma, \gamma'] \subset \mathcal{I}[\alpha, \beta]$  by using Proposition 3.18 and Lemma 3.38. For the general case, use induction on the distance between  $\gamma$  and  $\gamma'$  together with Lemma 3.38 and the previous observation to the pairs of vertices

$$\gamma \leq \gamma \wedge \gamma', \quad \gamma' \leq \gamma \wedge \gamma', \quad \gamma \vee \gamma' \leq \gamma, \quad \text{and} \quad \gamma \vee \gamma' \leq \gamma'$$

.

□

The following notion of orthant of a vertex is useful for us to examine the structure of  $\mathcal{C}_n^{(0)}$  carefully. For a vertex  $\alpha \in \mathcal{C}_n^{(0)}$ , orthant  $\mathcal{O}(\alpha)$  of  $\alpha$  is defined by

$$(3.13) \quad \mathcal{O}(\alpha) = \{\beta \mid \beta \geq \alpha\}.$$

Note that  $\mathcal{O}(\alpha)$  is also convex in  $\mathcal{C}_n^{(0)}$  for any vertex  $\alpha$ .

**Lemma 3.40.** *Suppose  $\mathcal{O}(\gamma) \cap \mathcal{I}[\alpha, \beta] \neq \emptyset$  for some  $\gamma \notin \mathcal{I}[\alpha, \beta]$ . There exists unique  $\delta_0 \in \mathcal{O}(\gamma) \cap \mathcal{I}[\alpha, \beta]$  with smallest distance from  $\gamma$ . Moreover  $d(\alpha, \gamma) = d(\alpha, \delta_0) + d(\delta_0, \gamma)$ .*

*Proof.* Suppose  $\gamma \notin \mathcal{I}[\alpha, \beta]$  and  $\mathcal{O}(\gamma) \cap \mathcal{I}[\alpha, \beta] = \{\delta_1, \dots, \delta_k\}$ . Since  $\gamma$  is a common lower bound of those elements, there exists *glb* of  $\{\delta_1, \dots, \delta_k\}$ . Let  $\delta_0$  denote the *glb*. The Claim 1 in proof of Theorem 3.41 shows that  $\delta_0$  is the unique vertex in  $\mathcal{I}[\alpha, \beta]$  with the smallest distance from  $\gamma$ . If  $d(\alpha, \gamma) > d(\alpha, \delta_0) + d(\delta_0, \gamma)$  then it is not difficult to show there exists another vertex  $\delta' \in \mathcal{O}(\gamma) \cap \mathcal{I}[\alpha, \beta]$  with smaller distance. So  $\delta_0$  satisfies the equality. □

**Theorem 3.41.**  $\mathcal{C}_n$  is a median graph for any  $n \in \mathbb{N}$ .

*Proof.* From Lemma 3.37, it suffices to show  $\mathcal{I}[\alpha, \beta] \cap \mathcal{I}[\beta, \gamma] \cap \mathcal{I}[\gamma, \alpha]$  is a singleton set for any  $\alpha, \beta$  and  $\gamma \notin \mathcal{I}[\alpha, \beta]$ .

**Case 1:**  $\mathcal{O}(\gamma) \cap \mathcal{I}[\alpha, \beta] \neq \emptyset$ . From Lemma 3.40, we see that there exists unique  $\delta_0 \in \mathcal{O}(\gamma)$  with smallest distance from  $\gamma$ . We want to show  $\mathcal{I}[\alpha, \beta] \cap \mathcal{I}[\beta, \gamma] \cap \mathcal{I}[\gamma, \alpha] = \{\delta_0\}$ . From Lemma 3.40 (the second assertion), a concatenation of two geodesics joining  $\alpha$  to  $\delta_0$  and  $\delta_0$  to  $\gamma$  is again a geodesic. Thus the intersection of three intervals contains  $\delta_0$ . Suppose the intersection contains another vertex  $w$ .

**Claim 1:**  $d(w, \gamma) > d(\delta_0, \gamma)$ . Say  $p_1, p_2, p_3$  are standard geodesics joining  $\alpha$  to  $\delta_0$ ,  $\alpha$  to  $w$  and  $w$  to  $\gamma$  respectively, given by

$$p_1 = \pi_1 \bar{\pi}_2, \quad p_2 = \pi'_1 \bar{\pi}'_2, \quad p_3 = \tau_1 \bar{\tau}_2,$$

and  $\delta_0 = \tau \gamma$  for some  $\pi_1, \pi_2, \pi'_1, \pi'_2, \tau_1, \tau_2, \tau \in \mathcal{T}_n$ . Figure 3.11 illustrates this situation for those paths  $p_1$  (solid red),  $p_2$  (solid blue) and  $p_3$  (solid black). Since the concatenation  $p_2 p_3$



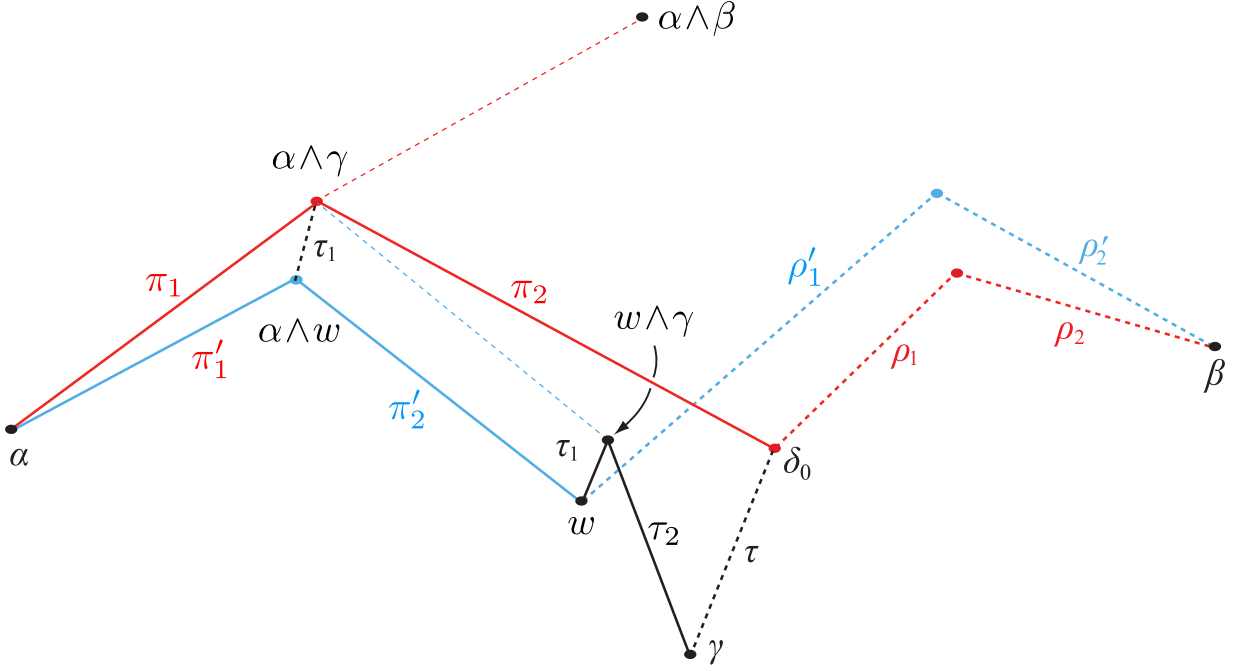


FIGURE 3.11.  $\delta_0 \in \mathcal{O}(\gamma)$  with  $d(w, \gamma) > d(\delta_0, \gamma)$

is a geodesic joining  $\alpha$  to  $\gamma$ , after applying liftings to  $p_2p_3$ , one obtains a standard geodesic passing  $\alpha \wedge \gamma$ . Consider the ascending path joining  $w$  to  $\alpha \wedge \gamma$  given by  $\tau_1\pi_2'$ . This passes  $w \wedge \gamma$  and so  $w \wedge \gamma \in \mathcal{I}[\alpha, \beta]$  by Corollary 3.39. Since  $w \wedge \gamma \geq \gamma$  we see that  $w \wedge \gamma \geq \delta_0$  and  $w \wedge \gamma \neq \delta_0$  by the definition of  $\delta_0$ . Thus  $d(w, \gamma) = d(w, w \wedge \gamma) + d(w \wedge \gamma, \gamma) > |\tau_2| > |\tau| = d(\delta_0, \gamma)$ .

**Claim 2:**  $w \notin \mathcal{I}[\alpha, \beta] \cap \mathcal{I}[\beta, \gamma] \cap \mathcal{I}[\gamma, \alpha]$ . Say  $p_4$  and  $p_5$  are standard geodesics joining  $\delta_0$  to  $\beta$ ,  $w$  to  $\beta$  respectively given by

$$p_4 = \rho_1\bar{\rho}_2, p_5 = \rho_1'\bar{\rho}_2'.$$

Since  $\delta_0$  belongs to the intersection the concatenation  $p_1p_4$  is a geodesic with length  $d(\alpha, \beta)$ . If  $w$  belongs to the intersection then the concatenation  $p_2p_5$  is a geodesic with the same length. So we must have

$$|\pi_1| + |\pi_2| + |\tau| = d(\alpha, \gamma) = |\pi_1'| + |\pi_2'| + |\tau_1'| + |\tau_2'|$$

and

$$|\pi_1| + |\pi_2| + |\rho_1| + |\rho_2| = |\pi_1'| + |\pi_2'| + |\rho_1'| + |\rho_2'|.$$

However claim 1 implies that  $|\tau_1'| + |\tau_2'| > |\tau|$  and hence  $|\pi_1| + |\pi_2| > |\pi_1'| + |\pi_2'|$ . So we have  $|\rho_1'| + |\rho_2'| > |\rho_1| + |\rho_2|$ . This means that any standard path of the concatenation  $\bar{p}_3p_5$  does not path  $\gamma \wedge \beta$ . So the path  $\bar{p}_3p_5$  is not a geodesic. Since paths  $p_3$  and  $p_5$  are arbitrary,  $w \notin \mathcal{I}[\gamma, \beta]$ .

**Case 2:**  $\mathcal{O}(\gamma) \cap \mathcal{I}[\alpha, \beta] = \emptyset$ . Obviously  $d(\mathcal{O}(\gamma), \mathcal{I}[\alpha, \beta]) = \min\{d(u, v) \mid u \in \mathcal{O}(\gamma), v \in \mathcal{I}[\alpha, \beta]\} > 0$ . Say the distance is  $d > 0$ . Let  $D \subset \mathcal{O}(\gamma)$  denote the set of all vertices of  $\mathcal{O}(\gamma)$  realizing  $d$ . Note that the cardinality of  $D$  is finite (there are finitely many elements in  $\mathcal{O}(\gamma)$  whose height is less than  $h(\alpha \wedge \beta)$  and, by the argument showing  $\delta_0$  is unique (Lemma

3.40), there are finitely many vertices realizing  $d$  whose height is greater than  $h(\alpha \wedge \beta)$ . Take  $glb$  over  $D$  and let  $\epsilon$  denote this unique element. Say  $\epsilon' \in \mathcal{I}[\alpha, \beta]$  is unique vertex with  $d(\epsilon', \epsilon) = d$  (uniqueness follows from the argument in the proof of Lemma 3.40). See Figure 3.12.

**Claim 3:**  $d(\alpha, \epsilon') + d + d(\epsilon, \gamma) = d(\alpha, \gamma)$  and  $d(\beta, \epsilon') + d + d(\epsilon, \gamma) = d(\beta, \gamma)$ . Pick any standard geodesics  $q_1 \bar{q}_2$  connecting  $\alpha$  to  $\epsilon'$  and ascending paths  $q_3$  and  $q_4$  joining  $\epsilon'$  to  $\epsilon$ , and  $\gamma$  to  $\epsilon'$  respectively. Then the concatenation  $q_1 \bar{q}_2 q_3 \bar{q}_4$  is a geodesic. Consider the standard path  $q_1 q_2 \bar{q}_3 \bar{q}_4$ . Observe that the ascending path  $q_1 q_2$  connecting  $\alpha$  to  $\alpha \wedge \gamma$  realizes the distance  $d(\alpha, \mathcal{O}(\gamma))$ . Thus any standard path given by  $q_1 q_2 \bar{q}_3 \bar{q}_4$  is a geodesic. Similar argument shows that  $d(\beta, \epsilon') + d + d(\epsilon, \gamma) = d(\beta, \gamma)$ . So  $\epsilon'$  belongs to the intersection  $\mathcal{I}[\alpha, \beta] \cap \mathcal{I}[\beta, \gamma] \cap \mathcal{I}[\gamma, \alpha]$ .

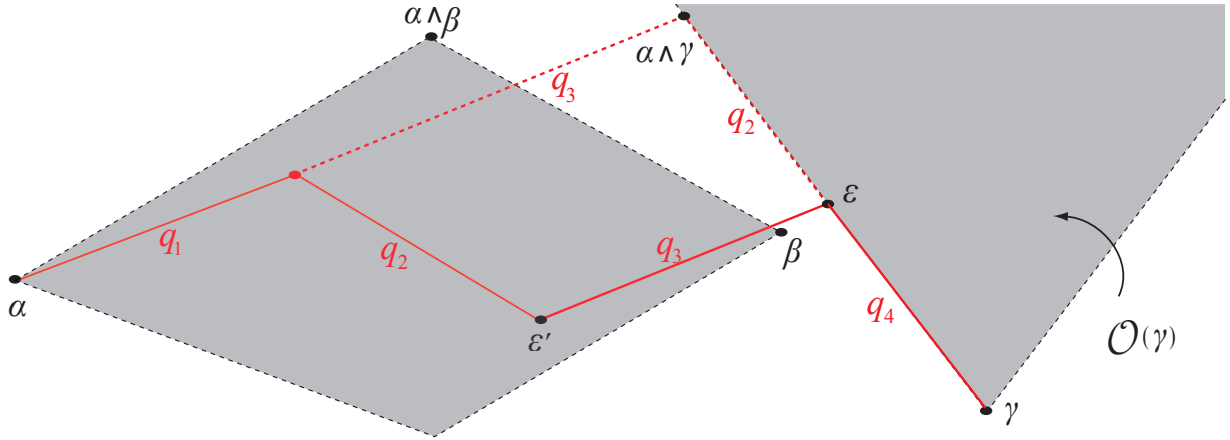


FIGURE 3.12. Vertices  $\epsilon$  and  $\epsilon'$  with  $d(\epsilon', \epsilon) = d(\mathcal{O}(\gamma), \mathcal{I}[\alpha, \beta])$

**Claim 4:**  $\mathcal{I}[\alpha, \beta] \cap \mathcal{I}[\beta, \gamma] \cap \mathcal{I}[\gamma, \alpha] = \{\epsilon'\}$ . Suppose the intersection contains another vertex  $w \in \mathcal{I}[\alpha, \beta]$ . We apply an analogous trick as in **Case 2:** with small difference with  $d(w, \gamma) > d + d(\epsilon, \gamma)$ . One can show this inequality together with

$$d(\alpha, \gamma) = d(\alpha, w) + d(w, \gamma)$$

yields

$$d(\beta, \gamma) > d(\beta, w) + d(w, \gamma).$$

□

**3.6. Properties of the action of  $\mathcal{H}_n$  on  $X_n$ .** In this section we examine the action of  $\mathcal{H}_n$  on  $X_n$ . First we examine that the stabilizer of every cell is a finite symmetric group.

**Lemma 3.42.** *Suppose  $\alpha$  is a vertex of  $X_n$  with  $h(\alpha) = h$ . Then the stabilizer of  $\alpha$  is the finite symmetric group  $\Sigma_h$  on  $h$  points.*

*Proof.* Let  $\Sigma_h \leq \mathcal{H}_n$  denote the symmetric group on a finite set  $S(\alpha) = Y_n - (Y_n)\alpha$ . We show the stabilizer of  $\alpha$  is simply  $\Sigma_h \leq \mathcal{H}_n$ . If  $g \in \Sigma$  then  $\alpha g = \alpha$ . Conversely, if  $g \in \mathcal{H}_n$  and  $\alpha g = \alpha$  then  $g$  restricted to the set  $(Y_n)\alpha$  must be the identity. So  $\text{supp}(g) \subset (\alpha)$  and hence  $g \in \Sigma_h$ . □

Recall the following definition of a Morse function defined on a (affine) CW-complex  $X$  where  $\varphi_j : \square^k \rightarrow \sigma_j^k \subset X^k$  denote the attaching map of  $k$ -cell  $\sigma_j^k$ .

**Definition 3.43 (Morse function).** A map  $f : X \rightarrow \mathbb{R}$  is a *Morse function* if

- for every cell  $\varphi_j(\square^k)$  of  $X$   $f\varphi_j : \square^k \rightarrow \mathbb{R}$  extends to an affine map  $\mathbb{R}^m \rightarrow \mathbb{R}$  and  $f\varphi_j$  is constant only when  $k = 0$  and
- the image of the 0-skeleton is discrete in  $\mathbb{R}$ .

Note that the map  $h$  defined on a cubing  $X_n$  has the property that  $h\varphi_j : \square^k \rightarrow \mathbb{R}$  extends to the standard height function  $\mathbb{R}^k \rightarrow \mathbb{R}$  up to the translation by  $h\varphi(0)$  (see Figure 3.4). Note also that  $h\varphi_j : \square^k \rightarrow \mathbb{R}$  is trivial only when  $k = 0$  for all  $j$ . Moreover, the image  $h(X_n^{(0)})$  is just  $h(\mathcal{C}_n) = h(\mathcal{M}_n) = \mathbb{Z}_{\geq 0}$ . Therefore the map  $h$  is a Morse function.

**Remark 3.44.** The map  $h : X_n \rightarrow \mathbb{R}_{\geq 0}$  satisfies the following

$$h(\alpha g) = h(\alpha) \text{ and } h(t\alpha) = h(t) + h(\alpha)$$

for all  $g \in \mathcal{H}_n$  and  $t \in \mathcal{T}_n$ . In that sense, the action of  $\mathcal{H}_n$  is ‘horizontal’ and the action of  $\mathcal{T}_n$  is ‘vertical’.

Let  $X_{n,r}$  denote the subcomplex of  $X_n$  consisting of cubes up to height  $r$ , i.e.,

$$X_{n,r} := \{\sigma \in X_n \mid h(\sigma) \subset [0, r]\}.$$

**Lemma 3.45.** Suppose a  $k$ -cube  $\sigma \subset X_n$  is generated by  $T = \{\tau_i, \dots, \tau_k\} \subset \{t_1, \dots, t_n\}$  with bottom vertex  $\alpha$ . There exists  $g \in \mathcal{H}_n$  such that  $\sigma \cdot g$  is the  $k$ -cube generated by  $T$  with bottom vertex  $t_1^{h(\alpha)}$ .

*Proof.* From the proof of Lemma 3.13, we see that there exists an element  $g \in \mathcal{H}_n$  such that  $\alpha g = t_1^{h(\alpha)}$ . Since the action of  $\mathcal{H}_n$  on  $X_n$  is cellular, a cube  $\sigma g$  is the desired cube bottom vertex  $t_1^{h(\alpha)}$ .  $\square$

**Corollary 3.46.** For  $r \in \mathbb{Z}_{\geq 0}$ ,  $\mathcal{H}_n$  acts on  $X_{n,r}$  cocompactly.

*Proof.* Fix  $r \in \mathbb{Z}_{\geq 0}$ . There are finitely many  $k$ -cubes with bottom vertex  $t_1^{r'}$  for  $1 \leq r' \leq r$  and  $0 \leq k \leq n$ . Lemma 3.45 implies that for any cube  $\sigma \subset X_{n,r}$  there exists  $g \in \mathcal{H}_n$  such that  $\sigma \cdot g$  is a  $k$ -cube with bottom vertex  $t_1^{r'}$  for some  $r' \leq r$ . So the quotient  $X_{n,r}/\mathcal{H}_n$  is finite.  $\square$

**Definition 3.47 (Semi-simple group action).** Let  $X$  be a metric space and let  $g$  be an isometry of  $X$ . The displacement function of  $g$  is the function  $d_g : X \rightarrow \mathbb{R}_{\geq 0}$  defined by  $d_g(x) = d(x, x.g)$ . The translation length of  $d_g$  is the number  $|d_g| := \inf\{d_g(x) \mid x \in X\}$ . The set of points where  $d_g$  attains this infimum will be denoted  $\text{Min}(d_g)$ . More generally, if  $G$  is a group acting by isometries on  $X$ , then  $\text{Min}(G) := \bigcap_{g \in G} \text{Min}(d_g)$ . An isometry  $d_g$  is called semi-simple if  $\text{Min}(d_g)$  is non-empty. An action of a group by isometries of  $X$  is called semi-simple if all of its elements are semi-simple.

The following theorem summarizes the action of  $\mathcal{H}_n$  on  $X_n$ , see Figure 3.13 for the underlying theme.

**Theorem A.** For each integer  $n \geq 1$ , there exists a  $n$ -dimensional cubing  $X_n$  and a Morse function  $h : X_n \rightarrow \mathbb{R}_{\geq 0}$  such that  $\mathcal{H}_n$  acts on  $X_n$  properly (but not cocompactly) by height-preserving semi-simple isometries. Furthermore, for each  $r \in \mathbb{R}_{\geq 0}$  the action of  $\mathcal{H}_n$  restricted to the level set  $h^{-1}(r)$  is cocompact.

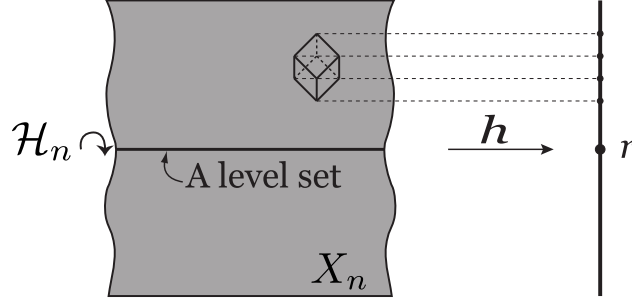


FIGURE 3.13. A Morse function  $h$  on a CAT(0) cubical complex  $X$

*Proof. Proper action.* We want show that, for  $\alpha, \beta \in X_n^{(0)}$  the set  $S(\alpha, \beta) := \{g \in \mathcal{H}_n | \alpha g = \beta\}$  is finite. An element  $\alpha$  has the right inverse (left inverse in composition of functions)  $\alpha^{-1} : (Y_n)\alpha \rightarrow Y_n$ . So if  $g \in S(\alpha, \beta)$  then  $g|_{(Y_n)\alpha}$  is completely determined by

$$g = \alpha^{-1}\beta.$$

This means that one can decompose  $g$  by

$$g = \begin{cases} \alpha^{-1}\beta & \text{on } (Y_n)\alpha \\ f & \text{on } S(\alpha) \end{cases}$$

for some  $f \in \mathcal{H}_n$  with  $\text{supp}(f) \subset S(\alpha)$ . Since there are finitely many  $f$  with  $\text{supp}(f) \subset S(\alpha)$ ,  $S(\alpha, \beta)$  is finite. Note that if  $\alpha = \beta$  then  $S(\alpha, \beta)$  is simply the finite symmetric group on  $h(\alpha) = h(\beta)$  points that we discussed in Lemma 3.42. Now suppose  $\sigma$  is a cube with vertices  $\{\alpha_1, \dots, \alpha_k\}$ . If  $\sigma g$  intersects  $\sigma$  non trivially then  $g \in S(\alpha_i, \alpha_j)$  for some  $1 \leq i, j \leq k$ . Therefore the action is proper.

**Height-preserving isometries.** The action preserves height by Remark 3.44. To show the action in question is by isometries, we want to use the fact that two actions of  $\mathcal{T}_n$  and  $\mathcal{H}_n$  on  $X_n$  commute. Suppose  $\alpha, \beta \in X_n^{(0)}$ . Consider the smallest convex set  $H$  containing  $\alpha$  and  $\beta$ , i.e., the intersection of all convex sets containing  $\alpha$  and  $\beta$ . Observe that the cell structure of  $X_n$  is completely determined by the action of  $\mathcal{T}_n$  on  $X_n^{(0)}$ . Since the two actions of  $\mathcal{T}_n$  and  $\mathcal{H}_n$  on  $X_n$  commute,  $Hg$  is a convex set with the same cell structure of  $H$ . By Corollary 3.32,  $\alpha$  and  $\beta$  are joined by the unique geodesic  $\ell$ , which lies in  $H$ . The image  $\ell g$  is the local geodesic (in  $Hg$ ) joining  $\alpha g$  and  $\beta g$ . Since  $Hg$  is convex  $\ell g$  is the global geodesic. So we have  $d(\alpha, \beta) = d(\alpha g, \beta g)$  for  $\alpha, \beta \in X_n^{(0)}$  and  $g \in \mathcal{H}_n$ .

**Semi-simple action.** For any element  $f \in \mathcal{H}_n$  with finite order, it is not difficult to check that  $d_f = 0$  and  $\text{Min}(d_g)$  is non empty. For example one can take  $\alpha \in \mathcal{M}_n$  such that  $\text{supp}(f) \subset S(\alpha)$  to see  $d(\alpha f, \alpha) = 0$  and  $\alpha \in \text{Min}(d_f)$ . Suppose  $g \in \mathcal{H}_n$  with  $\varphi(g) = (m_1, \dots, m_n) \neq \vec{0}$  ( $m_1 = -(m_2 + \dots + m_n)$ ).

**Claim:**  $d_g = \sqrt{\sum m_i^2}$ . We want to find explicit element  $\alpha \in X_n^{(0)}$  realizing  $d_g$ . The idea is to choose a vertex  $\alpha$  with big enough height so that ‘translation’ by  $g$  is realized clearly. Consider the element  $g' = g_1^{m_2} g_2^{m_3} \dots g_{n-1}^{m_n}$ . Since  $\varphi(g) = \varphi(g)'$ , there exists  $f \in \mathcal{H}_n$  such that  $g = fg'$  and that  $\text{supp}(f) \subset B_{n,r}$  where  $B_{n,r} \subset Y_n$  is the ball centered at the origin of radius  $r$ . Set  $r' = \max\{r, \sum |m_i|\}$  and consider the element  $\alpha = t_1^{r'} t_2^{r'} \dots t_n^{r'}$ . The maximality of  $r'$  ensures that  $\alpha, \alpha g' \in \mathcal{T}_n$ . By Corollary 3.16, there exists  $\alpha \vee \alpha g'$ . It is possible to find

explicit expression for  $\alpha \wedge \alpha g'$  as follows. Define  $k_i, k'_i \in \mathbb{Z}$ ,  $i = 1, \dots, n$  by

$$k_1 = \max\{0, -\sum_2^n m_i\}, \quad k'_1 = \max\{0, \sum_2^n m_i\} \text{ and}$$

$$k_i = \max\{0, m_i\}, \quad k_i = \max\{0, -m_i\} \text{ for } 2 \leq i \leq n.$$

Since  $\varphi(\alpha) = (r', \dots, r')$  and  $\varphi(\alpha g') = (r' + m_1, r' + m_2, \dots, r' + m_n)$ , the top vertex is given by

$$(3.14) \quad \tau\alpha = \alpha \wedge (\alpha g') = \tau'(\alpha g')$$

where  $\tau = t_1^{k_1} \dots t_n^{k_n}$  and  $\tau' = t_1^{k'_1} \dots t_n^{k'_n}$ . By Proposition 3.18,  $\alpha \vee \alpha g'$  satisfies

$$(3.15) \quad \alpha = \tau'(\alpha \vee \alpha g') \text{ and } \alpha g' = \tau(\alpha \vee \alpha g').$$

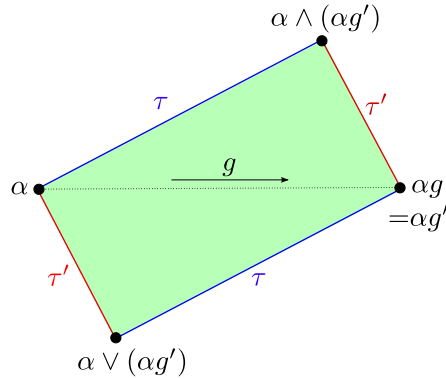


FIGURE 3.14. If  $h(\alpha)$  is big enough, the translation by  $g$  is realized in a ‘big rectangle’  $R$

Equalities (3.14) and (3.14) imply that there exists a ‘big rectangle’  $R$  which contains  $\alpha \wedge \alpha g'$  and  $\alpha \vee \alpha g'$  as its top and bottom vertices respectively (as described in Proposition 3.18). See Figure 3.14. Observe that  $R$  is the convex hull of two vertices  $\alpha$  and  $\alpha g'$ . The diagonal of  $R$  joining  $\alpha$  and  $\alpha g'$  has length  $\sqrt{\sum m_i^2}$ . The diagonal is the geodesic joining  $\alpha$  to  $\alpha g'$  because no smaller cube contains those two vertices. So we have

$$d(\alpha, \alpha g) = d(\alpha, \alpha f g') = d(\alpha, \alpha g') = \sqrt{\sum m_i^2}.$$

Note that the size of  $R$  is determined by  $\varphi(g) = (m_1, \dots, m_n)$  and that  $R$  has the smallest size among convex hulls containing  $\beta$  and  $\beta g$  for  $\beta \in X_n^{(0)}$ . This means  $d_g \geq d(\alpha, \alpha g)$ . Therefore  $d_g$  attains minimum at  $\alpha$  and  $\alpha \in \text{Min}(d_f)$ .

**Cocompact action.** Note that  $h^{-1}(r) \subset X_{n,r'}$  for  $r \in \mathbb{R}$  and  $r \leq r' \in \mathbb{Z}$ . By Remark 3.46, the action on the level set is cocompact for each  $r \in \mathbb{Z}_{\geq 0}$ .  $\square$

**3.7. Finiteness Properties of  $\mathcal{H}_n$ .** In this section we discuss finite properties of  $\mathcal{H}_n$ . We first recall properties  $F_n$  and  $FP_n$  (see [5],[8]) which are generalized concepts of being finitely generated and finitely presented of groups.

We say a group  $G = \langle \mathcal{A} \mid \mathcal{R} \rangle$  is *finitely generated* if  $\mathcal{A}$  is finite and *finitely presented* if both  $\mathcal{A}$  and  $\mathcal{R}$  are finite. These finite conditions can be interpreted topologically via *presentaion 2-complex (Cayley complex)*  $K = K(\mathcal{A}; \mathcal{R})$ .  $K$  has one vertex and it has one edge  $e_a^1$  (oriented

and labelled by  $a$ ) for each generator  $a \in \mathcal{A}$ . The 2-cells  $e_r^2$  of  $K$  are indexed by the relators  $r \in \mathcal{R}$ ; if  $r = a_1 \cdots a_k$  then  $\sigma_r$  is attached along the loop labelled by  $a_1 \cdots a_k$ . From the construction of  $K$  we see that it has finite 1-skeleton if  $G$  is finitely generated and it has finite 2-skeleton as well if  $G$  is finite presented. By the Seifert-Van Kampen theorem we have  $\pi_1(K) \cong G$ . Note that the existence of a complex  $K$  with finite 2-skeleton with  $\pi_1(K) \cong G$  guarantees finite presentedness of a group  $G$ .

More general finiteness properties are based on  $K(G, 1)$  spaces (*Eilenberg-Mac Lane complex*). For a group  $G$ , a complex  $K$  is called a  $K(G, 1)$  complex if  $\pi_1(K) \cong G$  and the universal cover  $\tilde{K}$  is contractible. It is known that for a group  $G$  there exists unique  $K(G, 1)$  complex having one vertex up to homotopy.

**Definition 3.48 (Property  $F_n$ ).** We say a group  $G$  has type  $F_n$  if there exists a  $K(G, 1)$  complex having finite  $n$ -skeleton.

Consider the augmented chain complex  $C_*(\tilde{K}; \mathbb{Z})$  of the universal cover of a  $K(G, 1)$  complex  $K$ .

$$\cdots \xrightarrow{\partial_3} C_2(\tilde{K}; \mathbb{Z}) \xrightarrow{\partial_2} C_1(\tilde{K}; \mathbb{Z}) \xrightarrow{\partial_1} C_0(\tilde{K}; \mathbb{Z}) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0.$$

This is unique free  $\mathbb{Z}G$ -resolution of  $\mathbb{Z}$  up to chain homotopy. Note that each  $C_i(\tilde{K}; \mathbb{Z})$  is finitely generated free  $\mathbb{Z}G$ -module if  $K$  has finite  $i$ -skeleton. A module is *projective* if it is a direct summand of a free module.

**Definition 3.49 (Property  $FP_n$ ).** We say a group  $G$  has type  $FP_n$  if there exists a projective  $\mathbb{Z}G$ -resolution of  $\mathbb{Z}$  which is finitely generated in dimensions  $\leq n$ .

**Remark 3.50.** An immediate consequence is that if a group  $G$  has type  $F_n$  then it has type  $FP_n$  for integers  $n \in \mathbb{N}$ .

A  $G$ -CW complex is a complex  $K$  together with a homomorphism  $G \rightarrow \text{Aut} K$ . A  $G$ -filtration of a CW complex  $K$  is a countable collection of  $G$ -subcomplexes  $K_0 \subset K_1 \subset \cdots$  such that  $K = \bigcup K_i$ .

Let  $K$  be a contractible  $G$ -CW complex which admits a  $G$ -filtration  $\{K_i\}$  satisfying

- the stabilizer of every cell is finitely presented and has type  $FP_n$  for all  $n$
- each  $K_i$  is finite mod  $G$
- for all sufficiently large  $j$ ,  $K_{j+1}$  is obtained from  $K_j$  by the adjunctions of  $n$ -cells, up to homotopy.

**Theorem 3.51 (Brown's Criterion).** *With the above assumption,  $G$  has type  $FP_{n-1}$  but not  $FP_n$ . If  $n \geq 3$  then  $G$  is finitely presented.*

Brown show the following finiteness properties of  $\mathcal{H}_n$  by constructing a CW complexes satisfying the above criterion ([6]). By Theorem A, a cubing  $X_n$ ,  $n \geq 1$ , satisfies the criterion above, Remark 3.46.

**Corollary B.** *For  $n \geq 2$ ,  $\mathcal{H}_n$  is of type  $FP_{n-1}$  but not  $FP_n$ , it is finitely presented for  $n \geq 3$ .*

*Proof.* We show our  $X_n$  satisfies the above Brown's criterion. By Corollary 3.32,  $X_n$  is contractible. Lemma 3.42 implies the first condition is satisfied since a finite symmetric group satisfies required finiteness properties. The second condition follows from Remark

3.46. With respect to a Morse function  $h : X_n \rightarrow \mathbb{R}$ , each descending link of a vertex  $v \in X_n$  is homotopic to bouquet of spheres  $\mathbb{S}^{n-1}$  if the height of  $h(v) \geq 2n - 1$  ( Lemma 3.52). This means that the passage from  $X_{n,h}$  to  $X_{n,h+1}$  consists of the adjunction of  $n$ -cubes, up to homotopy. The theorem therefore follows from the criterion Theorem 3.51.  $\square$

**Lemma 3.52.** *If  $h \geq 2n - 1$  then  $L_{n,h}$  homotopic to a bouquet of spheres  $\mathbb{S}^{n-1}$ .*

*Proof.* We argue by induction on  $n$ . If  $n = 1$ , then  $L_{n,h}$  is a bouquet of spheres  $\mathbb{S}^0$ , provided  $h \geq 2 \cdot 1 - 1$ . Assume  $L_{n-1,k}$  is homotopy equivalent to a bouquet of copies of  $\mathbb{S}^{n-2}$  if  $k \geq 2h - 1$  and  $n \geq 2$ .

We use Betvina-Brady Morse theory to understand the topology. Equip  $L = L_{n,h}$  with a Morse function  $f : L \rightarrow [0, 2] \subset \mathbb{R}$  as follows. First consider the partition of the vertex set of  $L_{n,h}$ :  $L_0 = \{(x, y) \mid x \geq 2, y \geq 2\} \cup \{(1, 1)\}$ ,  $L_1 = \{(1, y) \mid y \geq 2\}$  and  $L_2 = \{(x, 1) \mid x \geq 2\}$ . Arrange the vertices so that  $f(x, y) = h$  if  $(x, y) \in L_h$ ,  $h = 0, 1, 2$ . See Figure 3.15.

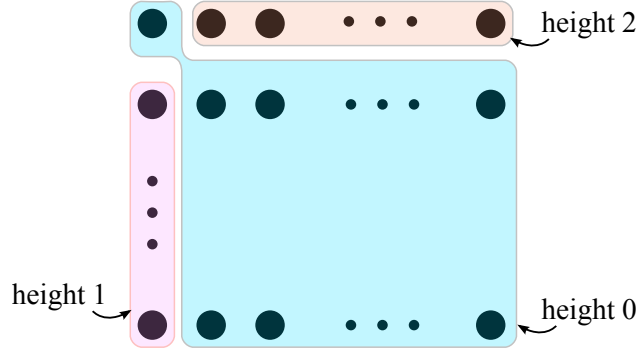


FIGURE 3.15. The complex  $L_{n,h}$  equipped with a Morse function  $f : L_{n,h} \rightarrow [0, 2]$

It is clear that  $f^{-1}(0)$  is a cone on  $L_{n-1,h-1}$  and so it is contractible. Note that even if there are horizontal cells at height 0 it is still true that  $f^{-1}[0, 1]$  is homotopy equivalent to  $f^{-1}(0)$  with the copies of  $Lk_{\downarrow}((1, y), L)$  coned off  $((1, y) \in L_1)$ . Observe that the descending link of a vertex  $(1, y) \in L_1$  (in  $L$ ) is spanned by vertices  $(x, y') \in L_0$  with  $y' \neq y$ . Now  $Lk_{\downarrow}((1, y), L) \simeq L_{n-1,h-2}$ ,  $f^{-1}[0, 1]$  is obtained from  $f^{-1}(0)$  by adjoining, for each  $(1, y)$ , a cone over  $L_{n-1,h-2}$ . In view of inductive hypothesis,  $f^{-1}[0, 1]$  is homotopy equivalent to a bouquet of spheres  $\mathbb{S}^{n-1}$ .

Observe that  $Lk_{\downarrow}((x, 1), L) \simeq L_{n-1,h-1}$  for each vertex  $(x, 1) \in L_2$ . So, by the inductive hypothesis again, the complex  $L = f^{-1}[0, 2]$  is homotopy equivalent to  $f^{-1}[0, 1]$  with the copies of spheres  $\mathbb{S}^{n-2}$  coned off. (Similar proof can be found in [6].)  $\square$

#### 4. ISOPERIMETRIC INEQUALITIES FOR $\mathcal{H}_n$

**4.1. Dehn functions of finitely presented groups.** We recall the (algebraic) definition of Dehn function of a group from [2]. Let  $P = \langle \mathcal{A} \mid \mathcal{R} \rangle$  be a finite presentation for a group  $G$  with the identity  $1_G$ . A word  $w$  is an element of the free monoid with the generating set  $\mathcal{A} \cup \mathcal{A}^{-1}$ . Denote the length of  $w$  by  $|w|_G$ . We say  $w$  is null-homotopic when  $w = 1_G$ , i.e.,  $w$  lies in the normal closure of  $\mathcal{R}$  in the free group  $F(\mathcal{A})$ . We define the *area* of a

null-homotopic word  $w$  to be

$$Area(w) := \min\{N \mid w =^{free} \prod_{i=1}^N u_i^{-1} r_i u_i \text{ with } u_i \in F(\mathcal{A}), r_i \in \mathcal{R}^{\pm 1}\}.$$

The *Dehn function* of  $P$  is the function  $\delta_P : \mathbb{N} \rightarrow \mathbb{N}$  defined by

$$\delta_P(x) := \max\{Area(w) \mid w = 1_G, |w|_G \leq x\}.$$

Although the Dehn function  $\delta_P(x)$  depends on the presentation  $P$ , asymptotic growth type of  $\delta_P(x)$  (as  $x$  tends to infinity) only depends on  $G$  up to the equivalence relation  $\simeq$  defined as follows (see [2]). Two functions  $f, g : \mathbb{N} \rightarrow [0, \infty)$  are said to be  $\simeq$  equivalent if  $f \preceq g$  and  $g \preceq f$ , where  $f \preceq g$  means that there exists a constant  $C > 0$  such that  $f(x) \leq Cg(Cx+C)+Cx+C$  for all  $x \in \mathbb{N}$ . Up to this equivalence relation, any finite presentation  $P$  of a group  $G$  determines the same asymptotic growth type, which is called the *Dehn function* of  $G$ . The Dehn function of  $G$  is denoted by  $\delta_G(x)$ . The Dehn function is an important invariant of group theory. If a group  $G$  is CAT(0), the  $\delta_G(x)$  is bounded above by  $x^2$ , see [4]. A group  $G$  is hyperbolic if and only if  $\delta_G(x) \simeq x$ , see [9]. In particular, every finite group has a linear Dehn function.

**Remark 4.1.** For any  $m \in \mathbb{N}$  the symmetric group  $\Sigma_m$  on  $\{1, 2, \dots, m\}$  satisfies  $\delta_{\Sigma_k}(x) \leq Cx + C$  for some  $C > 0$ . Note that the constant  $C$  depends on the group  $\Sigma_m$  and hence on  $m$ . We need an upper bound for  $\delta_{\Sigma_m}(x)$  which only depends on  $x$  (see Lemma 4.3).

**4.2. Exponential isoperimetric inequalities for  $\mathcal{H}_n$ .** In this section we aim to establish exponential upper bounds for  $\delta_{\mathcal{H}_n}$  for  $n \geq 3$ .

**Theorem D.** *For  $n \geq 3$ , the Dehn function  $\delta_{\mathcal{H}_n}(x)$  satisfies*

$$\delta_{\mathcal{H}_n}(x) \preceq e^x.$$

We start with the case  $n = 3$ . Let  $B_{3,x} \subset Y_3$  centered at the origin with radius  $x \in \mathbb{N}$ . Let  $\Sigma_{3,x} \leq \mathcal{H}_3$  denote the finite symmetric group on  $B_{3,x}$  given by (2.9). The following is an outline for the proof.

- (1) For a given word  $w = 1 \in \mathcal{H}_3$  with  $|w| \leq x$ , rewrite  $w$  by  $w' \in \Sigma_{3,x}$  by using a canonical way such that
  - $|w'|_{\Sigma_{3,x}} \leq x^5$ ,
  - the gap between  $w$  and  $w'$  is filled with area  $\leq x^2$ .
- (2) Establish a cubic upper bound for  $\delta_{\Sigma_{3,x}}(x)$ .
- (3) Bound  $Area_{\mathcal{H}_3}(r)$  for all relators  $r$  of  $\Sigma_{3,x}$  by  $e^x$ ,

Figure 4.1 illustrates the strategy. Now we have a desired upper bound for the area of  $w$  since

$$(4.1) \quad Area_{\mathcal{H}_3}(w) \leq (x^5)^3 \cdot e^{Ax+A} + x^2 \leq e^{(A+18)x+A} + x^2 \preceq e^x.$$

We remark that, in each step, any bound not exceeding exponential function is good enough for us. We have established exponential upper bound for  $\delta_{\mathcal{H}_3}(x)$  up to those claims.



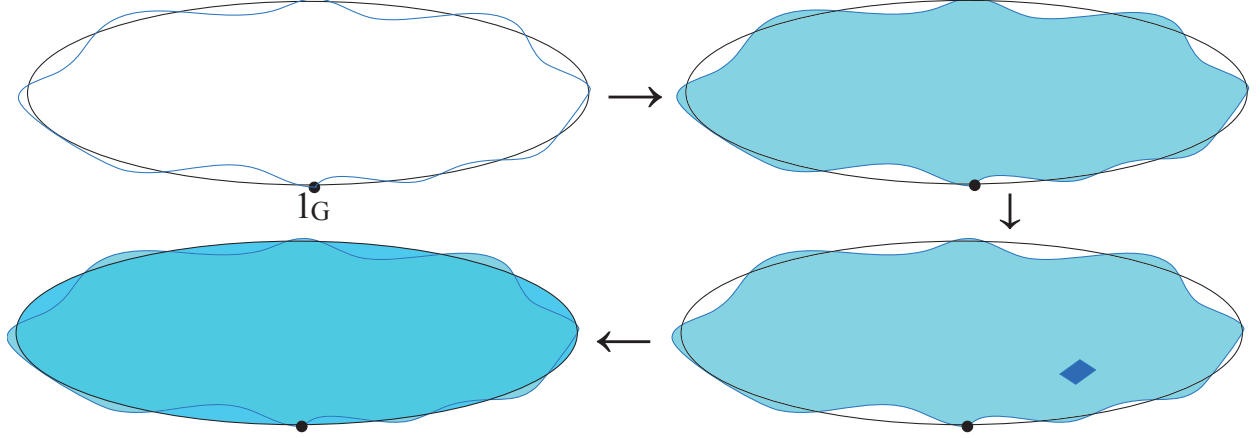


FIGURE 4.1. Sketch of the proof for  $\delta_{\mathcal{H}_3}(x) \preceq e^x$

First we establish a cubic upper bound for  $\delta_{\Sigma_{3,x}}(x)$  by using the following fact. Let  $S_m$  denote the finite symmetric group on  $m$  points with the Coxeter presentation given in (2.5), where  $\{\sigma_1, \dots, \sigma_{m-1}\}$  is the generating set. Let  $s_i = \sigma_{\phi(i)}$  for some  $\phi : \mathbb{N} \rightarrow \{1, 2, \dots, m-1\}$ .

**Theorem 4.2** (Deletion Theorem, [13]). *Let  $\Sigma_N$  denote the symmetric group on  $N$  letters with generators  $\sigma_1, \dots, \sigma_{N-1}$ . Suppose  $w = s_1 s_2 \dots s_k$ ,  $s_i \in \{\sigma_1, \dots, \sigma_{N-1}\}$ . If  $|w| < k$  then there exists  $i$  and  $j$  ( $1 \leq i < j \leq k$ ) such that*

$$(4.2) \quad s_{i+1} s_{i+2} \dots s_j = s_i s_{i+1} \dots s_{j-1},$$

and so

$$w = s_1 s_2 \dots \widehat{s_i} \dots \widehat{s_j} \dots s_k.$$

**Lemma 4.3.** *For any  $m \in \mathbb{N}$ , a null-homotopic word  $w \in \S_m$  with  $|w| \leq x$  satisfies  $\text{Area}(w) \leq x^3$ .*

*Proof.* We first show any word  $w \in \S_m$  given by the form

$$s_j = s_i^{s_{i+1} s_{i+2} \dots s_{j-1}}$$

as in (4.2) has at most quadratic area in  $|j - i|$ . ( $g^h := h^{-1} g h$ .) Suppose  $w = s_{k+1}(\bar{s}_0^{s_1 \dots s_k})$  represents the identity of  $\Sigma_m$ . We want to show  $\text{Area}(w) \leq k^2$  by induction on  $k$ .

The base case is obvious since  $s_2(\bar{s}_0^{s_1}) = 1$  is a single commutation relation. We may assume no two consecutive letters in  $w$  are the same. Now suppose  $w = s_{k+2}(\bar{s}_0^{s_1 \dots s_{k+1}})$  is null-homotopic. We consider two cases.

**Case 1.** There exists  $i_0$  ( $k+2 \geq i_0 \geq 1$ ) such that either

$$(4.3) \quad [s_{i_0}, s_i] = 1 \text{ for all } i_0 > i \geq 0$$

or

$$(4.4) \quad [s_{i_0}, s_i] = 1 \text{ for all } k+2 \geq i > i_0.$$

If the condition (4.3) holds then one can apply commutation relations consecutively to decompose the diagram of  $w$  into a  $s_{i_0}$ -corridor with length  $2(i_0 - 1) + 1$  and a diagram of  $w' = s_{k+1}(\bar{s}_0^{s_1 \dots \widehat{s_{i_0}} \dots s_k})$ . (See Figure 4.2) Thus, by induction assumption, we have

$$\text{Area}(w) \leq \text{Area}(w') + 2(i_0 - 1) + 1 < (k+1)^2.$$

By using an analogous argument together with condition (4.4), one can draw the same upper bound.

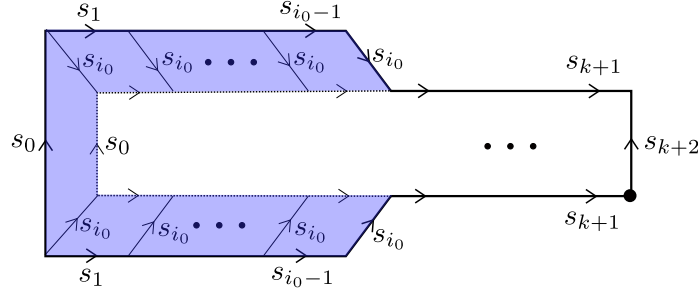


FIGURE 4.2. The diagram for  $w$  can be reduced by commutation relations in Case 1.

**Case 2.** For each  $k+1 \geq i \geq 1$  there exists  $i'' > i > i'$  such that

$$(4.5) \quad [s_{i''}, s_i] \neq 1 \text{ and } [s_i, s_{i'}] \neq 1.$$

The above condition simply says that  $\sigma(i'') = \sigma(i) \pm 1$  and  $\sigma(i') = \sigma(i) \pm 1$ . We need to examine subwords of  $w$ . Let  $w_j$  denote the subword  $\bar{s}_0^{s_1 \cdots s_j}$  for  $j = 1, \dots, k+1$ . For  $j = 0$ ,  $w_0$  is simply defined to be  $s_0$ . Note that each  $w_j$  is a transposition since it is a conjugation of a transposition  $s_0$ . For a transposition  $s \in \Sigma_m$  let  $d_+(s)$  and  $d_-(s)$  denote the two points of  $\text{supp}(s)$  with  $d_+(s) > d_-(s)$ . The function  $d : \{0, 1, \dots, k+1\} \rightarrow \mathbb{N}$  measures the difference  $d_+ - d_-$ , i.e.,  $d(j) = d_+(w_j) - d_-(w_j)$  for  $j = 0, 1, \dots, k+1$ . Set  $D(j) = \{d_-(w_j), d_-(w_j) + 1, \dots, d_+(w_j)\}$ . Note that  $d(0) = d(k+1) = 1$ . So there exists  $i$  such that  $d(i+1) \leq d(i)$ , say  $i_0$  is the smallest such number. Observe that  $|d(j) - d(j+1)| \leq 1$  for all  $j$  since conjugating by  $s_j$  introduces at most one point to  $D(j-1)$ . The following observation is crucial to establish desired bound;

$$d(w_{i_0}) = i_0 + 1.$$

This identity implies there is 1-to-1 correspondence between

$$(4.6) \quad D(i_0) - D(0) \text{ and } \{s_1, s_2, \dots, s_{i_0}\}.$$

This bijection and the condition (4.5) implies

$$(4.7) \quad d_+(s_i) = d_+(s_j) + 1 \Rightarrow i > j$$

for all  $i$  with  $i_0 \leq i \leq 1$ .

From the 1-to-1 correspondence (4.6) and the fact that  $D(s_{i_0+1}) \subset D(w_{i_0})$ , there exists unique  $i_0 > i' \geq 0$  such that  $s_{i_0+1} = s_{i'}$ . Say  $s_{i_0+1} = s_{i'} = (p \ p+1)$ , transposition exchanging  $p$  and  $p+1$ . On the other hand, if  $d_-(s_{i_0+1}) \geq d_+(s_0)$  then, from (4.7), we see that there exists unique  $i''$  with  $i_0 \geq i'' > i'$  such that  $s_{i''} = (p+1 \ p+2)$ . Observe that  $s_{i_0+1}$  commutes with  $s_i$  for all  $i$  with  $i_0 \geq i > i''$  and that  $s_{i'}$  commute with  $s_j$  for all  $j$  with  $i'' > j \geq i'$ . Apply those commutation relations to rearrange letters in the expression of  $w = s_{k+2}(\bar{s}_0^{s_1 \cdots s_{k+1}})$  so that  $s_{i_0+1}$ ,  $s_{i''}$  and  $s_{i'}$  show up in a row. Then apply the relation

$$s_{i_0+1} s_{i''} s_{i'} = s_{i'} s_{i''} s_{i'} = s_{i''} s_{i'} s_{i''},$$

where the second identity comes from the braid relation  $((p+1)(p+1)(p+2))^3 = 1$ . Now one applies the argument of **Case 1** since  $s_{i''}$  commute with all  $s_i$  for  $i \leq i''$ . In all, from

$$\begin{aligned} s_1 \cdots s_{i_0} s_{i_0+1} &= s_1 \cdots s_{i'} \cdots s_{i''} \cdots s_{i_0} s_{i_0+1} \\ &= s_1 \cdots \widehat{s_{i'}} \cdots s_{i''-1} (s_{i'} s_{i''} s_{i_0+1}) s_{i''+1} \cdots s_{i_0} \\ &= s_1 \cdots \widehat{s_{i'}} \cdots s_{i''-1} (s_{i''} s_{i'} s_{i''}) s_{i''+1} \cdots s_{i_0} \\ &= s_{i''} s_1 \cdots \widehat{s_{i'}} \cdots s_{i''-1} (s_{i'} s_{i''}) s_{i''+1} \cdots s_{i_0} \end{aligned}$$

we have

$$w = s_{k+2}(\bar{s}_0^{s_1 \cdots s_{i''} \cdots s_{k+1}}) = s_{k+2}(\bar{s}_0^{s_{i''} s_1 \cdots \widehat{s_{i'}} \cdots s_{k+1}}) = s_{k+2}(\bar{s}_0^{s_1 \cdots \widehat{s_{i'}} \cdots s_{k+1}})$$

The number of required relators in the above process is at most

$$2\{(i_0 - i'' - 1) + (i'' - i' - 1) + 1 + i''\} + k^2 \leq k^2 + 2k < (k+1)^2$$

since  $i_0 \leq \frac{k}{2}$ . So we have shown any word in the form of (4.2) has area at most  $|j - i|^2 \leq k^2$ .

This means whenever one applies the identity (4.2) to a null-homotopic word  $s_1 \cdots s_x \in \Sigma_m$ , the number of relators is bounded by  $x^2$ . At the same time one can reduce the number of generators in the expression by two. So we have  $\delta_{\Sigma_m}(x) \leq x^3$ .  $\square$

As discussed in Remark 2.3, one of important feature of  $\mathcal{H}_n$  is  $\Sigma_{n,\infty} \hookrightarrow \mathcal{H}_n$  when  $n \geq 3$ . If a word  $w \in \mathcal{H}_3$  is null-homotopic then  $w \in \Sigma_{3,\infty} = \cup_r \Sigma_{3,r}$  (Lemma 2.11) It is natural to ask the minimum  $r$  so that  $w \in \Sigma_{3,r}$ . We have a reasonable bound for  $r$  as well as the length of  $w$  in  $\Sigma_{3,r}$ .

**Lemma 4.4.** *There is a canonical way so that any null-homotopic word  $w \in \mathcal{H}_3$  with  $|w| \leq x$  can be written as a word  $w' \in \Sigma_{3,x}$  with length at most  $O(x^5)$ .*

*Proof.* Suppose  $w = 1_{\mathcal{H}_3}$  with  $|w|_{\mathcal{H}_3} = x$ . Then  $w$  can be written as  $w = g_1^{m_1} g_2^{n_1} \alpha^{\epsilon_1} \cdots g_1^{m_k} g_2^{n_k} \alpha^{\epsilon_k}$  for some  $m_i, n_i \in \mathbb{Z}$ ,  $\epsilon_i \in \{-1, 0, 1\}$ .

Use the identity

$$(4.8) \quad g\alpha \equiv \alpha^{g^{-1}}g$$

to rewrite  $w$  as

$$(4.9) \quad w \equiv f g_1^{m_1} g_2^{n_1} \cdots g_1^{m_k} g_2^{n_k},$$

where  $f$  is a product of at most  $k(< x)$  many transpositions whose supports belong to the ball  $B_{3,x}$  of radius  $x$ . The symbol  $\equiv$  indicates that no relators are required in the above rewriting. The identity in free group  $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1} = \alpha$  allows one to exchange  $g_1$  and  $g_2$ ;

$$(4.10) \quad g_2 g_1 = \alpha^{-1} g_1 g_2, \quad g_2 g_1^{-1} = \alpha^{g_1} g_1^{-1} g_2,$$

$$(4.11) \quad g_2^{-1} g_1 = \alpha^{g_1 g_2 g_1^{-1}} g_1 g_2^{-1}, \quad \text{and } g_2^{-1} g_1^{-1} = (\alpha^{-1})^{g_2 g_1} g_1^{-1} g_2^{-1}.$$

Use identities (4.10) and (4.11) together with (4.8) to rewrite (4.9) again as

$$(4.12) \quad w = f f_1 f_2 \cdots f_\ell g_1^0 g_2^0$$

where each  $f_i$  is a transposition of  $B_{3,x}$ . By applying identities (4.10) and (4.11) consecutively one can show that  $g_2^{n_i} g_1^{m_j}$  can be written as  $f' g_1^{m_j} g_2^{n_i}$  and that  $f'$  is a product of at most

$|n_i m_j|$  transpositions of  $B_{3,x}$ . This means that the number  $\ell$  in expression (4.12) is bounded by

$$(\sum |m_j|)(\sum |n_i|) \leq x^2.$$

We need to bound the length of transpositions  $f_i$ 's in (4.12),  $i = 1, \dots, \ell$ . The isomorphism  $\chi^* : \Sigma_{3,x} \cong S_{3x}$  (Theorem 2.10) transforms each  $f_i \in \Sigma_{3,x}$  into a transposition  $\chi^*(f_i)$  of the set  $\{1, \dots, 3x\}$ . By Lemma 4.6,  $\chi^*(f_i)$  has length at most  $O(x^2)$  in  $S_{3x}$ . Since each generator of  $S_{3x}$  becomes a word of  $S_{3x}$  of length  $\leq 2x$  under  $\chi^*$ ,  $\chi^*(f_i)$  correspond to a word of  $\Sigma_{3,x}$  with length at most  $O(x^3)$ . In all, the element  $f f_1 f_2 \dots f_\ell$  in the expression (4.12) has length at most  $O(x^5)$ .

Finally we need to calculate the area between two loops, namely one given by  $w$  and the other one given by  $w'$ . Let  $B > 0$  denote the maximum number of relations that we used in (4.10) and (4.11). Note that whenever one exchanges  $g_1$  and  $g_2$  by applying an identity of (4.10) and (4.11), a transposition is produced. We already calculated the number of those exchanges of two generators, which is bounded by  $x^2$ .  $\square$

The argument if the proof of the above lemma extends to general cases  $n \geq 4$ .

**Lemma 4.5.** *There is a canonical way so that any null-homotopic word  $w \in \mathcal{H}_n$  with  $|w| \leq x$  can be written as a word in  $\Sigma_{n,x}$  with length at most  $O(x^5)$ .*

*Proof.* Identities (4.8)(4.10)(4.11) holds for  $g \in \mathcal{H}_n$  and for all pair of generators  $g_i, g_j$  of  $\mathcal{H}_n$ . Moreover the isomorphism between  $\Sigma_{3,x}$  and  $S_{3x}$  (Theorem 2.10) allows one to bound the length of each transposition of  $B_{n,r}$  by  $O(x^3)$ . As before we establish an upper bounds:  $O(x^5)$  for the length of rewritten word  $w'$ , and  $x^2$  for the area between two loops  $w$  and  $w'$ .  $\square$

The following fact can be found in [13].

**Lemma 4.6.** *Let  $S_m$  be the finite symmetric group with the Coxeter system. For  $\sigma \in S_m$ , set  $r_i(\sigma) = |\{j : i < j \text{ but } \sigma(i) > \sigma(j)\}|$ . The length of an element  $\sigma$  is given by  $\sum_1^m r_i(\sigma)$ . In particular, there exist a unique element with the largest length  $\sum_1^{m-1} i$ .*

**Lemma 4.7.** *Any relator  $R$  of  $\Sigma_{3,r}$  requires at most  $O(e^x)$  relators of  $\mathcal{H}_n$ .*

*Proof.* Each generator of  $\Sigma_{3,r}$  is an involution. Observe that this can be expressed as appropriate conjugation of  $\alpha^2 = 1$  in  $\mathcal{H}_3$ . Similarly a braid relator  $(\sigma_i \sigma_{i+1})^3 = 1$  can be expressed by conjugation of the relator  $(\alpha \alpha^{g_1})^3$ . Thus each relator of two types of  $\Sigma_{3,r}$  requires area only 1 in  $\mathcal{H}_n$ . For commutation relator of  $S_{n,r}$  we need to examine two sequences of words  $u_k = \alpha^{\bar{g}_1^k} \alpha^{\bar{g}_2^k}$  and  $v_k = [\alpha, \alpha^{\bar{g}_1^{k+1}}]$ . Note that we already check  $u_k$  and  $u_k$  represent the identity in Lemma 2.15. One can use simultaneous induction on  $k$  together with the argument in the proof of Lemma 2.15 to show areas of  $u_k$  and  $v_k$  are bounded above by  $O(e^x)$ . The fillings depicted in Figure 4.3 require  $O(e^x)$  relators. Since one can produce all commutation relation of  $\Sigma_{3,r}$  by taking appropriate conjugation of  $v_k$ , all the relators of  $\Sigma_{3,r}$  require at most  $O(e^x)$  relators of  $\mathcal{H}_n$ .  $\square$

*Proof of Theorem D.* Fix  $n \geq 4$ . We follow the 3-step plan that we applied to  $\mathcal{H}_3$ . For Step 1, Lemma 4.5 provides the same upper bounds:  $O(x^5)$  for the length of rewritten word, and  $O(x^2)$  for the gap between two words. The upper bound established in Lemma 4.3 does not depend on the subscript. This means we still have the same cubic upper bound

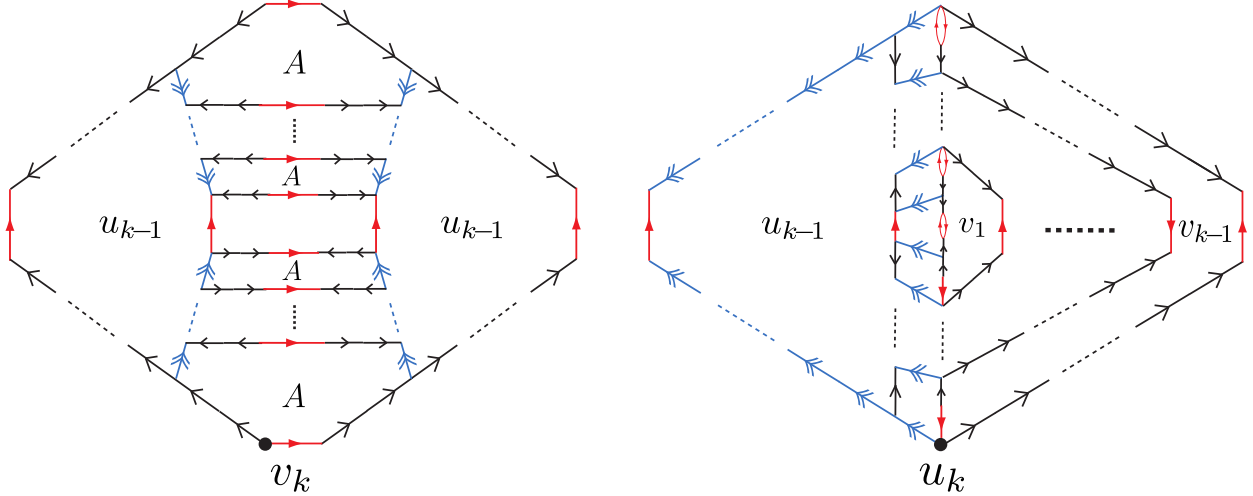


FIGURE 4.3. Diagrams of  $u_k = \alpha^{\bar{g}_1^k} \alpha^{\bar{g}_2^k}$  and  $v_k = [\alpha, \alpha^{\bar{g}_1^{k+1}}]$  ( $A : \alpha^{g_1^{-2}g_2} = \alpha^{g_1^{-2}}$ )

for  $\delta_{\Sigma_{n,x}}(x)$ . One can show first two types of relations of  $\Sigma_{n,x}$  require area 1 by taking appropriate conjugations as before. Note that there are variations of words  $u_k$  and  $v_k$  in  $\mathcal{H}_n$ . Instead of using  $g_1$  and  $g_2$  one can use  $g_i$  and  $g_i$  ( $1 \leq i < j \leq n-1$ ) to generate commutation relations of  $\Sigma_{n,x}$ . Again, exponential upper bounds for those words can be established by simultaneous induction on the length of two sequences of words. In all, we establish exponential upper bound for  $\mathcal{H}_n$ .

**Remark 4.8.** We remark that if words  $u_k$  and  $v_k$  (and their variations for general cases) have exponential areas then exponential upper bounds for  $\delta_{\mathcal{H}_3}$  ( $\delta_{\mathcal{H}_3}$ ) is sharp.

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